1. Introduction

This document provides the theoretical basis for computing the static deformation due to a point source in a plane layered isotropic medium. The discussion starts with the infinite medium solution for the dynamic medium response for point and multipole sources in cartesian coordinates, followed by an expression of the solution in cylindrical coordinates. Next the static solution in cylindrical coordinates for a step source time function is derived by taking the limit $\omega \rightarrow 0$. This solution permits the derivation of the displacement-stress discontinuities across the source depth which are required to solve the transformed differential equations. Using various integrals of Bessel functions, the static solution is described in terms of the cylindrical coordinate system. Next propagator matrices are developed from the homogeneous solution of the differential equation governing the z-dependence.

As a further test, the problem of displacement at the surface of a halfspace is derived both in the wavenumber and in the spatial domains as a tool for validating the multilayered propagator matrix solution.

Finally we will use the resulting programs in a realistic test case.

2. Infinite medium solution

Consider the local (x_1, x_2, x_3) coordinate system in Figure 1, with x_1 north and x_2 east and x_3 down. Alternatively this may be represented as an (x, y, z) coordinate system.

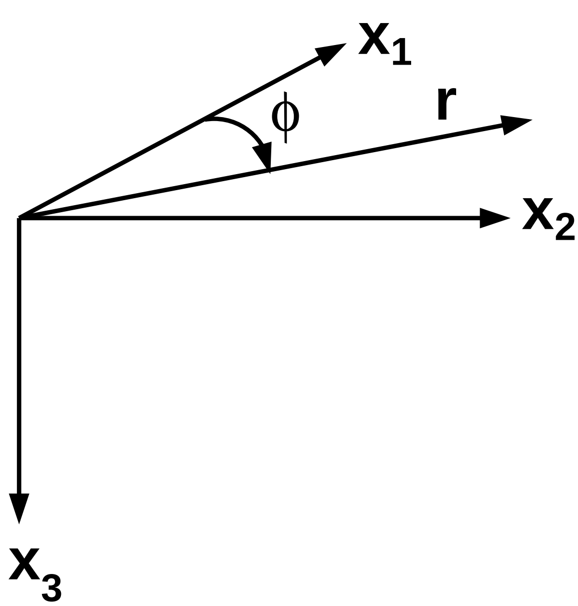

Fig. 1. Coordinate system for development of the theory. Note that the the convention for measurement at the surface of the earth is that z is positive upward. Thus the vertical component of any computed waveform will be opposite in sign to that derived theoretically for the z-positive downward coordinte system, e.g., x_3.

In the frequency domain, the elastic medium response to a point force in space and with an impulsive in time function is given by the relation

$$u_i = G_{ij} F_j \tag{5.3.18}$$

where G_{ij} is the Fourier transform of the *i'th* component of displacement for a unit force in the $j'th$ direction and the F_j are the Fourier transforms of the point forces acting in the $j-$ direction. The point force acts at the orgin of the coordinate system. The G_{ij} are given by

$$4\pi\rho G_{ij}(\mathbf{x},\omega) = (3\gamma_i\gamma_j - \delta_{ij})\exp(-i\omega R/\alpha)/(-\omega^2 R^3)$$
$$+(3\gamma_i\gamma_j - \delta_{ij})\exp(-i\omega R/\alpha)/(i\omega\alpha R^2)$$
$$+\gamma_i\gamma_j \exp(-i\omega R/\alpha)/(\alpha^2 R)$$
$$-(3\gamma_i\gamma_j - \delta_{ij})\exp(-i\omega R/\beta)/(-\omega^2 R^3) \quad (5.3.19)$$
$$-(3\gamma_i\gamma_j - \delta_{ij})\exp(-i\omega R/\beta)/(i\omega\beta R^2)$$
$$-\gamma_i\gamma_j \exp(-i\omega R/\beta)/(\beta^2 R)$$
$$+\delta_{ij}\exp(-i\omega R/\beta)/(\beta^2 R)$$

where $R^2 = x_1^2 + x_2^2 + x_3^2$, and the direction cosine is defined as $\gamma_j = x_j/R$. The compressional- and shear-wave velocities are indicted by α and β, respectively, and the density by ρ.

If the force applied is step-like, then the displacement traces in the time domain have a step offset at times following the S-wave arrival. This is illustrated in Figure 2. These offsets are the permanent deformation due to the application of a step force.

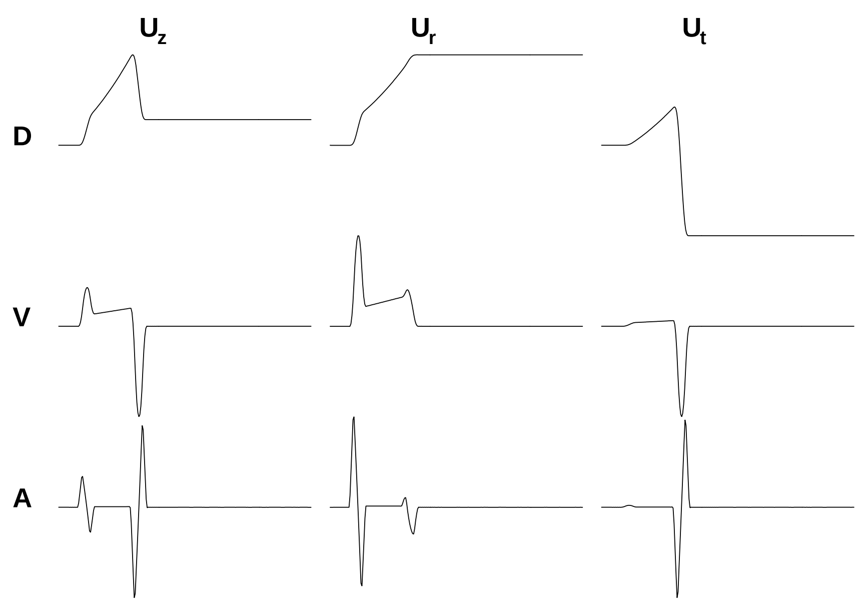

Fig. 2. Infinite medium response to a point force in the x_1 direction at a location (14.14, 14.14, 0.0) from a source located at (0,0, 5). In the figure U_z is in the up-direction, U_r and U_ϕ are positive in the direction of increasing r and ϕ, respectively.

This offset can easily be obtained by taking the limit of (5.3.19) as $\omega \to 0$. The reason for this is very simple. Let $d(t)$ be the displacement as a function of t, which is related to the velocity, $v(t)$, by the integral $d(t) = \int_{-\infty}^{t} v(\tau) d\tau$. The Fourier transform of the velocity time history, $V(\omega) = \int_{-\infty}^{\infty} v(\tau) e^{-i\omega\tau} d\tau$, at $\omega = 0$ is $V(0) = \int_{-\infty}^{\infty} v(\tau) d\tau$. Thus we see that $d(\infty) = V(0)$.

If the applied force, $F_j(t)$, is a step like function, e.g., $\lim_{t\to\infty} F_j(t) = F_{0j}$, then $F_{0j} = \lim_{\omega\to 0} \omega F_j(\omega)$. In the simplest case, consider the simple time function, $F_{0j} U(t)$. Using the Fourier transform of this step force, the Fourier transform of the ground velocity

$$i\omega u_i = i\omega G_{ij} F_{0j} \left(\pi \delta(\omega) + \frac{1}{i\omega} \right)$$

Taking the limit $\omega \to 0$, we see that

$$d(\infty) = V(0) = G_{ij} F_{0j}$$

The static displacement for a point step force with a step time function in the j direction is thus

$$4\pi\rho g_{ij} = \frac{1}{2R} (\gamma_i \gamma_j - \delta_{ij}) (\frac{1}{\beta^2} - \frac{1}{\alpha^2}) + \frac{\delta_{ij}}{\beta^2 R}$$

Note that the distance dependence of this permanent deformation is $O(R^{-1})$.

Next we consider the isotropic elastic medium response for the couple and dipole sources. The Fourier transform of the displacement is related to the moment tnesor elements through the relation

$$u_i = G_{ij,k} M_{jk}$$

In this expression, the notation $G_{ij,k}$ means $-\partial G_{ij}/\partial x_k$ where the x_k is the coordinate of the observastion point. The $G_{ij,k}$ is expressly given by the relation

$$\begin{aligned} 4\pi\rho(i\omega)^2 G_{ij,k} = &-(3\gamma_i\delta_{jk} + 3\gamma_j\delta_{ik} + 3\gamma_k\delta_{ij} - 15\gamma_i\gamma_j\gamma_k) \exp(-i\omega R/\alpha) (1/R^4) \\ &- (3\gamma_i\delta_{jk} + 3\gamma_j\delta_{ik} + 3\gamma_k\delta_{ij} - 15\gamma_i\gamma_j\gamma_k) \exp(-i\omega R/\alpha) (1/R^3) (i\omega/\alpha) \\ &- (\gamma_i\delta_{jk} + \gamma_j\delta_{ik} + \gamma_k\delta_{ij} - 6\gamma_i\gamma_j\gamma_k) \exp(-i\omega R/\alpha) (1/R^2) (i\omega/\alpha)^2 \\ &+ \gamma_i\gamma_j\gamma_k \exp(-i\omega R/\alpha) (1/R) (i\omega/\alpha)^3 \qquad (5.4.6) \\ &+ (3\gamma_i\delta_{jk} + 3\gamma_j\delta_{ik} + 3\gamma_k\delta_{ij} - 15\gamma_i\gamma_j\gamma_k) \exp(-i\omega R/\beta) (1/R^4) \\ &+ (3\gamma_i\delta_{jk} + 3\gamma_j\delta_{ik} + 3\gamma_k\delta_{ij} - 15\gamma_i\gamma_j\gamma_k) \exp(-i\omega R/\beta) (1/R^3) (i\omega/\beta) \\ &+ (\gamma_i\delta_{jk} + \gamma_j\delta_{ik} + 2\gamma_k\delta_{ij} - 6\gamma_i\gamma_j\gamma_k) \exp(-i\omega R/\beta) (1/R^2) (i\omega/\beta)^2 \end{aligned}$$

$$- (\gamma_i \gamma_j \gamma_k - \gamma_k \delta_{ij}) \exp(-i\omega R/\beta) (1/R) (i\omega/\beta)^3$$

Figure 3 shows the three component displacement, velocity and acceleration for a step-like M_{jk} function. We see a permanent offset in displacement, which does not seem to be as distinct as for a step-like point force.

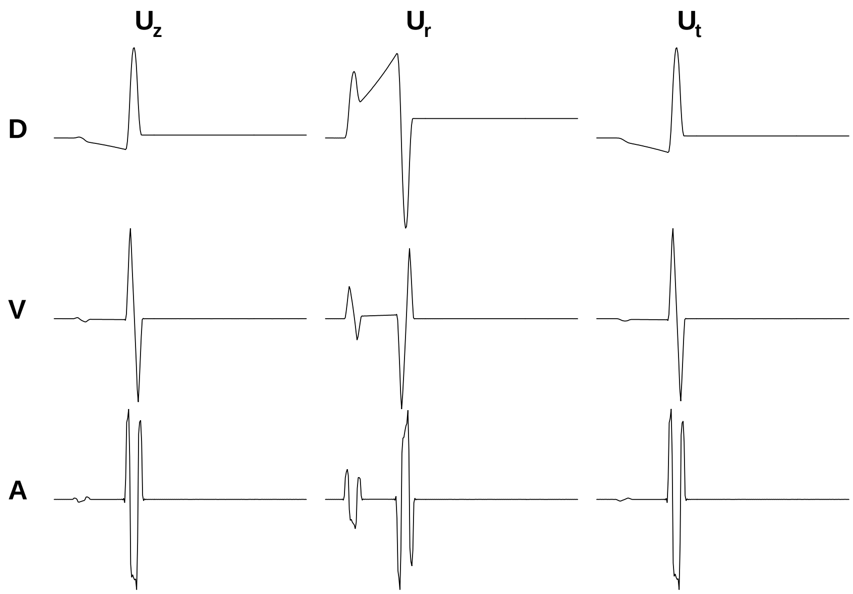

Fig. 3. Infinite medium response to a moment tensor force corresponding to a shear disloction with strike 45°, dip 45° and rake 45°. acting at a location (14.14, 14.14, 0.0) from a source located at (0,0, 10). In the figure U_z is in the up-direction.

Using the same procedure as used for the point force, the static displacement field due to a step change in the point source moment tensor elements is

$$u_k = g_{k,ij} M_{0ij}$$

where

$$4\pi \rho g_{k,ij} = \left(\gamma_i \delta_{jk} + \gamma_j \delta_{ik} + \gamma_k \delta_{ij} - 3\gamma_i \gamma_j \gamma_k \right) \frac{1}{2\alpha^2 R^2}$$

$$- \left(\gamma_i \delta_{jk} + \gamma_j \delta_{ik} - \gamma_k \delta_{ij} - 3\gamma_i \gamma_j \gamma_k \right) \frac{1}{2\alpha^2 R^2}$$

Note that the static displacement falls of with distance as $1/R^2$, which is faster than the falloff for the step force solution.

The synthetic seismogram of the Computer Programs in Seismology package were developed to generate waveforms in the time domain for sources and receivers in a plane layered structure. For isotropic and transverse-isotropic media, the solution is expressed in a cylindrical cooordinate system. The cylindrical coordinate system, (r, ϕ, z) and the (x_1, x_2, x_3) coordinate systems are related by the relations

$$x_1 = r \cos \phi$$

$$x_2 = r \sin \phi$$

$$x_3 = z$$

and the displacements are related by

$$\begin{bmatrix} u_r \\ u_\phi \\ u_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

In the cylindrical coordinate system, the displacements, (u_r, u_ϕ, u_z) for an arbitrary point force and moment tensor are given by the following relations.

$$u_z(r, z, h, \omega) = (F_1 \cos\phi + F_2 \sin\phi) ZHF + F_3 ZVF$$
$$+ M_{11}\left[\frac{ZSS}{2}\cos 2\phi - \frac{ZDD}{6} + \frac{ZEX}{3} \right]$$
$$+ M_{22}\left[\frac{-ZSS}{2}\cos 2\phi - \frac{ZDD}{6} + \frac{ZEX}{3} \right]$$
$$+ M_{33}\left[\frac{ZDD}{3} + \frac{ZEX}{3} \right]$$
$$+ M_{12}\left[ZSS \sin 2\phi \right] \qquad (5.6.8a)$$
$$+ M_{13}\left[ZDS \cos\phi \right]$$
$$+ M_{23}\left[ZDS \sin\phi \right]$$

$$u_r(r, z, h, \omega) = (F_1 \cos\phi + F_2 \sin\phi) RHF + F_3 RVF$$
$$+ M_{11}\left[\frac{RSS}{2}\cos 2\phi - \frac{RDD}{6} + \frac{REX}{3} \right]$$
$$+ M_{22}\left[\frac{-RSS}{2}\cos 2\phi - \frac{RDD}{6} + \frac{REX}{3} \right]$$
$$+ M_{33}\left[\frac{RDD}{3} + \frac{REX}{3} \right]$$
$$+ M_{12}\left[RSS \sin 2\phi \right] \qquad (5.6.8b)$$
$$+ M_{13}\left[RDS \cos\phi \right]$$

$$+ M_{23}\left[\ RDS\ \sin\phi\right]$$

$$u_\phi(r, z, h, \omega) = (+F_1 \sin\phi - F_2 \cos\phi)THF$$

$$+ M_{11}\left[\frac{TSS}{2}\ \sin 2\phi\right]$$

$$+ M_{22}\left[\frac{-TSS}{2}\ \sin 2\phi\right]$$

$$+ M_{12}\left[-TSS\ \cos 2\phi\right] \qquad (5.6.8c)$$

$$+ M_{13}\left[TDS\ \sin\phi\right]$$

$$+ M_{23}\left[-TDS\ \cos\phi\right].$$

In (5.6.8) the functions, *ZDD*, *RDD*, etc., depend on the source and and receiver positions, h and z, respectively, the cylindrical distance r and angular frequency ω. The azimuthal dependence is given by the trigonometric terms involving ϕ. The individual Green's functions are defined as integrals over wavenumber as follow:

$$ZDD = \int_0^\infty F_1(k,\omega)J_0(kr)kdk \qquad (5.6.10a)$$

$$RDD = -\int_0^\infty F_2(k,\omega)J_1(kr)kdk \qquad (5.6.10b)$$

$$ZDS = \int_0^\infty F_3(k,\omega)J_1(kr)kdk \qquad (5.6.10c)$$

$$RDS = \int_0^\infty F_4(k,\omega)J_0(kr)kdk \qquad (5.6.10d)$$

$$-\frac{1}{r}\int_0^\infty [F_4(k,\omega)\ +\ F_{13}(k,\omega)]\ J_1(kr)dk$$

$$TDS = \int_0^\infty F_{13}(k,\omega)J_0(kr)kdk \qquad (5.6.10e)$$

$$- \frac{1}{r} \int_0^\infty [F_4(k,\omega) \; + \; F_{13}(k,\omega)] \, J_1(kr) dk$$

$$ZSS = \int_0^\infty F_5(k,\omega) J_2(kr) k dk \tag{5.6.10f}$$

$$RSS = \int_0^\infty F_6(k,\omega) J_1(kr) k dk \tag{5.6.10g}$$

$$- \frac{2}{r} \int_0^\infty [F_6(k,\omega) \; + \; F_{14}(k,\omega)] \, J_2(kr) dk$$

$$TSS = \int_0^\infty F_{14}(k,\omega) J_1(kr) k dk \tag{5.6.10h}$$

$$- \frac{2}{r} \int_0^\infty [F_6(k,\omega) \; + \; F_{14}(k,\omega)] \, J_2(kr) dk$$

$$ZEX = \int_0^\infty F_7(k,\omega) J_0(kr) k dk \tag{5.6.10i}$$

$$REX = - \int_0^\infty F_8(k,\omega) J_1(kr) k dk \tag{5.6.10j}$$

$$ZVF = \int_0^\infty F_9(k,\omega) J_0(kr) k dk \tag{5.6.10k}$$

$$RVF = - \int_0^\infty F_{10}(k,\omega) J_1(kr) k dk \tag{5.6.10l}$$

$$ZHF = \int_0^\infty F_{11}(k,\omega) J_1(kr) k dk \tag{5.6.10m}$$

$$RHF = \int_0^\infty F_{12}(k,\omega) J_0(kr) k dk \tag{5.6.10n}$$

$$- \frac{1}{r} \int_0^\infty [F_{12}(k,\omega) \; + \; F_{15}(k,\omega)] \, J_1(kr) dk$$

$$THF = \int_0^\infty F_{15}(k,\omega) J_0(kr) kdk \tag{5.6.10o}$$

$$- \frac{1}{r} \int_0^\infty [F_{12}(k,\omega) \ + \ F_{15}(k,\omega)] \, J_1(kr) dk$$

The integrands for the isotropic wholespace problem are

$$F_1(k,\omega) = - \frac{1}{4\pi\rho(i\omega)^2} [(2k_\alpha^2 - 3k^2) e^{-\nu_\alpha |z|} + 3k^2 e^{-\nu_\beta |z|}] \ sgn(z) \tag{5.6.12a}$$

$$F_2(k,\omega) = \frac{k}{4\pi\rho(i\omega)^2} [(2k_\alpha^2 - 3k^2) \frac{e^{-\nu_\alpha |z|}}{\nu_\alpha} + 3\nu_\beta e^{-\nu_\beta |z|}] \tag{5.6.12c}$$

$$F_3(k,\omega) = \frac{k}{4\pi\rho(i\omega)^2} [2\nu_\alpha e^{-\nu_\alpha |z|} - (2k^2 - k_\beta^2) \frac{e^{-\nu_\beta |z|}}{\nu_\beta}] \tag{5.6.12c}$$

$$F_4(k,\omega) = - \frac{1}{4\pi\rho(i\omega)^2} [2k^2 e^{-\nu_\alpha |z|} - (2k^2 - k_\beta^2) e^{-\nu_\beta |z|}] \ sgn(z) \tag{5.6.12d}$$

$$F_5(k,\omega) = \frac{k^2}{4\pi\rho(i\omega)^2} [e^{-\nu_\alpha |z|} - e^{-\nu_\beta |z|}] \ sgn(z) \tag{5.6.12e}$$

$$F_6(k,\omega) = - \frac{k}{4\pi\rho(i\omega)^2} [\frac{k^2}{\nu_\alpha} e^{-\nu_\alpha |z|} - \nu_\beta e^{-\nu_\beta |z|}] \tag{5.6.12f}$$

$$F_7(k,\omega) = - \frac{1}{4\pi\rho(i\omega)^2} k_\alpha{}^2 e^{-\nu_\alpha |z|} \ sgn(z) \tag{5.6.12g}$$

$$F_8(k,\omega) = \frac{k}{4\pi\rho(i\omega)^2 \nu_\alpha} k_\alpha{}^2 e^{-\nu_\alpha |z|} \tag{5.6.12h}$$

$$F_9(k,\omega) = \frac{1}{4\pi\rho(i\omega)^2} [\nu_\alpha e^{-\nu_\alpha |z|} - \frac{k^2}{\nu_\beta} e^{-\nu_\beta |z|}] \tag{5.6.12i}$$

$$F_{10}(k,\omega) = - \frac{k}{4\pi\rho(i\omega)^2} [e^{-\nu_\alpha |z|} - e^{-\nu_\beta |z|}] \ sgn(z) \tag{5.6.12j}$$

$$F_{11}(k,\omega) = \frac{k}{4\pi\rho(i\omega)^2} [e^{-\nu_\alpha |z|} - e^{-\nu_\beta |z|}] \ sgn(z) \tag{5.6.12k}$$

$$F_{12}(k,\omega) = - \frac{1}{4\pi\rho(i\omega)^2} [\frac{k^2}{\nu_\alpha} e^{-\nu_\alpha |z|} - \nu_\beta e^{-\nu_\beta |z|}] \tag{5.6.12l}$$

$$F_{13}(k,\omega) = \frac{1}{4\pi\rho(i\omega)^2} k_\beta{}^2 e^{-\nu_\beta |z|} \ sgn(z) \tag{5.6.12m}$$

$$F_{14}(k,\omega) = \frac{k}{4\pi\rho(i\omega)^2}\,\frac{k_\beta^2}{\nu_\beta}\,e^{-\nu_\beta|z|} \quad (5.6.12n)$$

$$F_{15}(k,\omega) = \frac{1}{4\pi\rho(i\omega)^2}\,\frac{k_\beta^2}{\nu_\beta}\,e^{-\nu_\beta|z|} \quad (5.6.12o)$$

where function *sgn(z)* is defined as

$$sgn(z) = \begin{cases} 0 & z < 0 \\ 1 & z \ge 0 \end{cases}.$$

3. The halfspace problem

The solution of the wave propagation problem for a buried source and receiver in a uniform isotropic halfspace can be solve using matrix solution of a boundary value problem or constructed in terms of individuausing components. To set the framework for the solution, consider Figure 4. Figure 4a illustrates the wholespace problem just considered. In this case the solution (5.6.12) consists of terms sech as $e^{-\nu_\alpha|z-h|}$ and $e^{-\nu_\beta|z-h|}$, which indicate P- and S-wave signals that have propagated the vertical distance $|z-h|$. The other terms in (5.6.12) represent the source excitation and radiation pattern.

Figure 4b conceptually represents the SH-problem when a free surface is introduced. In addition to the direct wave there will be reflected SH wave which travels a total vertical distance of $z+h$. In addition to a different radiation term from the source the reflection coefficient at the free surface must be considered. Finally Figure 4c considers the P-SV problem for which the observed signal will have both P- and SV- components, and the P- and SV- signals indicent on the free surface can be converted on reflection to the other wave type. In this case the observed wavefield will have 6 terms.

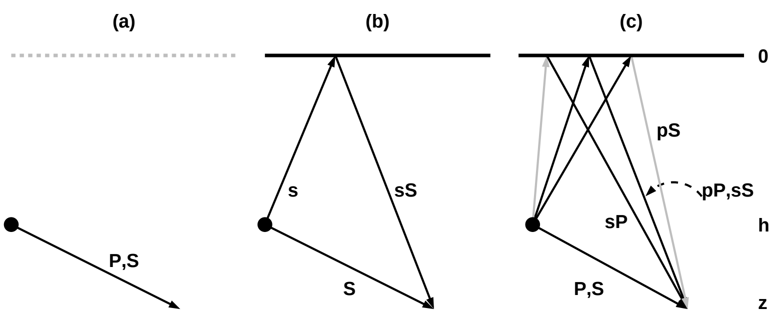

Fig. 4. Ray diagrams for wave propagation. (a) wholespace problem; (b) halfspace SH problem; (c) halfspace P-SV problem.

After careful work, the expressions in the integrams of (5.6.10) obtained. These equations are written such that direct-wave term of (5.6.12) can be readily identified.

$$F_1 = \frac{1}{4\pi\rho(i\omega)^2}\left[\frac{2k_\alpha^2 - 3k^2}{\nu_\alpha}\left(-e^{-\nu_\alpha|h-z|}P_Z + e^{-\nu_\alpha(h+z)}R_{PP}P_Z^D + e^{-\nu_\alpha h-\nu_\beta z}R_{PS}S_Z^D\right)\right.$$

$$+3k\left(-sgn(z-h)e^{-\nu_\beta|h-z|}S_Z + e^{-\nu_\beta h-\nu_\alpha z}R_{SP}P_Z^D + e^{-\nu_\beta(h+z)}R_{SS}S_Z^D\right)\Bigg]$$

$$F_2 = \frac{1}{4\pi\rho(i\omega)^2}\left[\frac{2k_\alpha^2-3k^2}{\nu_\alpha}\left(e^{-\nu_\alpha|h-z|}P_R + e^{-\nu_\alpha(h+z)}R_{PP}P_R^D + e^{-\nu_\alpha h-\nu_\beta z}R_{PS}S_R^D\right)\right.$$

$$\left.+3k\left(-sgn(z-h)e^{-\nu_\beta|h-z|}S_R + e^{-\nu_\beta h-\nu_\alpha z}R_{SP}P_R^D + e^{-\nu_\beta(h+z)}R_{SS}S_R^D\right)\right]$$

$$F_3 = \frac{1}{4\pi\rho(i\omega)^2}\left[2k\left(-sgn(z-h)e^{-\nu_\alpha|h-z|}P_Z + e^{-\nu_\alpha(h+z)}R_{PP}P_Z^D + e^{-\nu_\alpha h-\nu_\beta z}R_{PS}S_Z^D\right)\right.$$

$$\left.-\frac{2k^2-k_\beta^2}{\nu_\beta}\left(e^{-\nu_\beta|h-z|}S_Z + e^{-\nu_\beta h-\nu_\alpha z}R_{SP}P_Z^D + e^{-\nu_\beta(h+z)}R_{SS}S_Z^D\right)\right]$$

$$F_4 = \frac{1}{4\pi\rho(i\omega)^2}\left[2k\left(-sgn(z-h)e^{-\nu_\alpha|h-z|}P_R + e^{-\nu_\alpha(h+z)}R_{PP}P_R^D + e^{-\nu_\alpha h-\nu_\beta z}R_{PS}S_R^D\right)\right.$$

$$\left.-\frac{2k^2-k_\beta^2}{\nu_\beta}\left(e^{-\nu_\beta|h-z|}S_R + e^{-\nu_\beta h-\nu_\alpha z}R_{SP}P_R^D + e^{-\nu_\beta(h+z)}R_{SS}S_R^D\right)\right]$$

$$F_5 = \frac{1}{4\pi\rho(i\omega)^2}\left[-\frac{k^2}{\nu_\alpha}\left(e^{-\nu_\alpha|h-z|}P_Z + e^{-\nu_\alpha(h+z)}R_{PP}P_Z^D + e^{-\nu_\alpha h-\nu_\beta z}R_{PS}S_Z^D\right)\right.$$

$$\left.+k\left(-sgn(z-h)e^{-\nu_\beta|h-z|}S_Z + e^{-\nu_\beta h-\nu_\alpha z}R_{SP}P_Z^D + e^{-\nu_\beta(h+z)}R_{SS}S_Z^D\right)\right]$$

$$F_6 = \frac{1}{4\pi\rho(i\omega)^2}\left[-\frac{k^2}{\nu_\alpha}\left(e^{-\nu_\alpha|h-z|}P_R + e^{-\nu_\alpha(h+z)}R_{PP}P_R^D + e^{-\nu_\alpha h-\nu_\beta z}R_{PS}S_R^D\right)\right.$$

$$\left.+k\left(-sgn(z-h)e^{-\nu_\beta|h-z|}S_R + e^{-\nu_\beta h-\nu_\alpha z}R_{SP}P_R^D + e^{-\nu_\beta(h+z)}R_{SS}S_R^D\right)\right]$$

$$F_7 = \frac{k_\alpha^2}{4\pi\rho(i\omega)^2\nu_\alpha}\left[e^{-\nu_\alpha|h-z|}P_Z + e^{-\nu_\alpha(h+z)}R_{PP}P_Z^D + e^{-\nu_\alpha h-\nu_\beta z}R_{PS}S_Z^D\right]$$

$$F_8 = \frac{k_\alpha^2}{4\pi\rho(i\omega)^2\nu_\alpha}\left[e^{-\nu_\alpha|h-z|}P_R + e^{-\nu_\alpha(h+z)}R_{PP}P_R^D + e^{-\nu_\alpha h-\nu_\beta z}R_{PS}S_R^D\right]$$

$$F_9 = \frac{1}{4\pi\rho(i\omega)^2}\left[(-sgn(z-h))e^{-\nu_\alpha|h-z|}P_Z + e^{-\nu_\alpha(h+z)}R_{PP}P_Z^D + e^{-\nu_\alpha h-\nu_\beta z}R_{PS}S_Z^D\right.$$

$$- \frac{k}{\nu_\beta} (e^{-\nu_\beta |h-z|} S_Z + e^{-\nu_\beta (h+z)} R_{SS} S_Z^D + e^{-\nu_\beta h - \nu_\alpha z} R_{SP} P_Z^D) \Big]$$

$$F_{10} = \frac{1}{4\pi \rho (i\omega)^2} \Big[(-sgn(z-h)) e^{-\nu_\alpha |h-z|} P_R + e^{-\nu_\alpha (h+z)} R_{PP} P_R^D + e^{-\nu_\alpha h - \nu_\beta z} R_{PS} S_R^D$$

$$- \frac{k}{\nu_\beta} (e^{-\nu_\beta |h-z|} S_R + e^{-\nu_\beta (h+z)} R_{SS} S_R^D + e^{-\nu_\beta h - \nu_\alpha z} R_{SP} P_R^D) \Big]$$

$$F_{11} = \frac{1}{4\pi \rho (i\omega)^2} \Big[- \frac{k}{\nu_\alpha} (e^{-\nu_\alpha |h-z|} P_Z + e^{-\nu_\alpha (h+z)} R_{PP} P_Z^D + e^{-\nu_\alpha h - \nu_\beta z} R_{PS} S_Z^D)$$

$$-sgn(z-h) e^{-\nu_\beta |h-z|} S_Z + e^{-\nu_\beta h - \nu_\alpha z} R_{SP} P_Z^D + e^{-\nu_\beta (h+z)} R_{SS} S_Z^D \Big]$$

$$F_{12} = \frac{1}{4\pi \rho (i\omega)^2} \Big[- \frac{k}{\nu_\alpha} (e^{-\nu_\alpha |h-z|} P_R + e^{-\nu_\alpha (h+z)} R_{PP} P_R^D + e^{-\nu_\alpha h - \nu_\beta z} R_{PS} S_R^D)$$

$$-sgn(z-h) e^{-\nu_\beta |h-z|} S_R + e^{-\nu_\beta h - \nu_\alpha z} R_{SP} P_R^D + e^{-\nu_\beta (h+z)} R_{SS} S_R^D \Big]$$

$$F_{13} = \frac{k_\beta^2}{4\pi \rho (i\omega)^2 k} \Big[-sgn(z-h) e^{-\nu_\beta |h-z|} S_H - e^{-\nu_\beta (h+z)} R_{ShSh} S_H^D \Big]$$

$$F_{14} = \frac{k_\beta^2}{4\pi \rho (i\omega)^2 \nu_\beta} \Big[e^{-\nu_\beta |h-z|} S_H + e^{-\nu_\beta (h+z)} R_{ShSh} S_H^D \Big]$$

$$F_{15} = \frac{k_\beta^2}{4\pi \rho (i\omega)^2 k \beta \nu_\beta} \Big[e^{-\nu_\beta |h-z|} S_H + e^{-\nu_\beta (h+z)} R_{ShSh} S_H^D \Big]$$

The free surface P-SV reflection coefficients are

$$R_{PP} = - \frac{(\gamma - 1)^2 + \gamma^2 \nu_\alpha \nu_\beta / k^2}{F_R}$$

$$R_{PS} = - \frac{2\gamma(\gamma - 1) \nu_\alpha / k}{F_R}$$

$$R_{SP} = - \frac{2\gamma(\gamma - 1) \nu_\beta / k}{F_R}$$

$$R_{SS} = - \frac{(\gamma - 1)^2 + \gamma^2 \nu_\alpha \nu_\beta / k^2}{F_R}$$

where $\gamma = 2k^2/k_\beta^2$ and $F_R = (\gamma - 1)^2 - \gamma^2 \nu_\alpha \nu_\beta / k^2$.
The free surface reflection coefficient for SH is simply

$$R_{ShSh} = 1.$$

The conversion of a wave type to vector displacement is given by the terms $P_R = k$, P_Z^D is $-\nu_\alpha$, $P_Z = -sgn(z-h)\nu_\alpha$, $S_Z = k$ and $S_{RD} = -\nu_\beta$ and $S_R = -sgn(z-h)\nu_\beta$.
Motion at the top of the halfspace.

The motion at the top of the halfspace is obtained by setting the receiver depth to $z = 0$ in the expression above to give the following expressions:

$$F_1 = -\frac{2}{4\pi\rho(i\omega)^2 F_R}\left[(2k_\alpha^2 - 3k^2)(\gamma - 1)e^{-\nu_\alpha h} + 3\gamma\nu_\alpha\nu_\beta e^{-\nu_\beta h}\right]$$

$$F_2 = -\frac{2}{4\pi\rho(i\omega)^2 F_R}\left[(2k_\alpha^2 - 3k^2)\gamma \frac{\nu_\beta}{k} e^{-\nu_\alpha h} + 3k(\gamma - 1)\nu_\beta e^{-\nu_\beta h}\right]$$

$$F_3 = -\frac{4k(\gamma - 1)\nu_\alpha}{4\pi\rho(i\omega)^2 F_R}\left[e^{-\nu_\alpha h} - e^{-\nu_\beta h}\right]$$

$$F_4 = -\frac{2}{4\pi\rho(i\omega)^2 F_R}\left[2\gamma\nu_\alpha\nu_\beta e^{-\nu_\alpha h} - k_\beta^2(\gamma - 1)^2 e^{-\nu_\beta h}\right]$$

$$F_5 = \frac{2}{4\pi\rho(i\omega)^2 F_R}\left[k^2(\gamma - 1) e^{-\nu_\alpha h} - \gamma\nu_\alpha\nu_\beta e^{-\nu_\beta h}\right]$$

$$F_6 = \frac{2k\nu_\beta}{4\pi\rho(i\omega)^2 F_R}\left[\gamma e^{-\nu_\alpha h} - (\gamma - 1) e^{-\nu_\beta h}\right]$$

$$F_7 = \frac{-2k_\alpha^2(\gamma - 1)}{4\pi\rho(i\omega)^2 F_R} e^{-\nu_\alpha h}$$

$$F_8 = -\frac{2\gamma k_\alpha^2 \nu_\beta}{4\pi\rho(i\omega)^2 k F_R} e^{-\nu_\alpha h}$$

$$F_9 = -\frac{2\nu_\alpha}{4\pi\rho(i\omega)^2 F_R}\left[(\gamma - 1) e^{-\nu_\alpha h} - \gamma e^{-\nu_\beta h}\right]$$

$$F_{10} = -\frac{2}{4\pi\rho(i\omega)^2 F_R}\left[\gamma \frac{\nu_\alpha \nu_\beta}{k} e^{-\nu_\alpha h} - k(\gamma - 1) e^{-\nu_\beta h}\right]$$

$$F_{11} = \frac{2}{4\pi\rho(i\omega)^2 F_R}\left[k(\gamma - 1) e^{-\nu_\alpha h} - \gamma\nu_\alpha \frac{\nu_\beta}{k} e^{-\nu_\beta h}\right]$$

$$F_{12} = \frac{2\nu_\beta}{4\pi\rho(i\omega)^2 F_R}\left[\gamma e^{-\nu_\alpha h} - (\gamma - 1) e^{-\nu_\beta h}\right]$$

$$F_{13} = \frac{-2k_\beta^2}{4\pi\rho(i\omega)^2}\, e^{-\nu_\beta h}$$

$$F_{14} = \frac{2kk_\beta^2}{4\pi\rho(i\omega)^2\nu_\beta}\, e^{-\nu_\beta h}$$

$$F_{15} = \frac{2k_\beta^2}{4\pi\rho(i\omega)^2\nu_\beta}\, e^{-\nu_\beta h}$$

To derive these simpler relations we used the relations

$$P_Z^U + R_{PP}\, P_Z^D + R_{PS}\, S_Z^D = -\, 2(\gamma - 1)\nu_\alpha / F_R$$

$$P_R^U + R_{PP}\, P_R^D + R_{PS}\, S_R^D = -\, 2\gamma \nu_\alpha \nu_\beta / kF_R$$

$$S_Z^U + R_{SP}\, P_Z^D + R_{SS}\, S_Z^D = -\, 2\gamma \nu_\alpha \nu_\beta / kF_R$$

$$S_R^U + R_{SP}\, P_R^D + R_{SS}\, S_R^D = -\, 2(\gamma - 1)\nu_\beta / F_R$$

Figures 5 and 6 present the response to a point force and point moment tensor source, respectively, for a buried source and receiver at the surface of the halfspace. The dustinguishing feature of the time series is the greater complexity due to an arrival between P and S on the verttical and radial components, which is due to an S wave leaving the source upward and then refracted as P along the surface. There is also the Rayleigh wave onn these components.

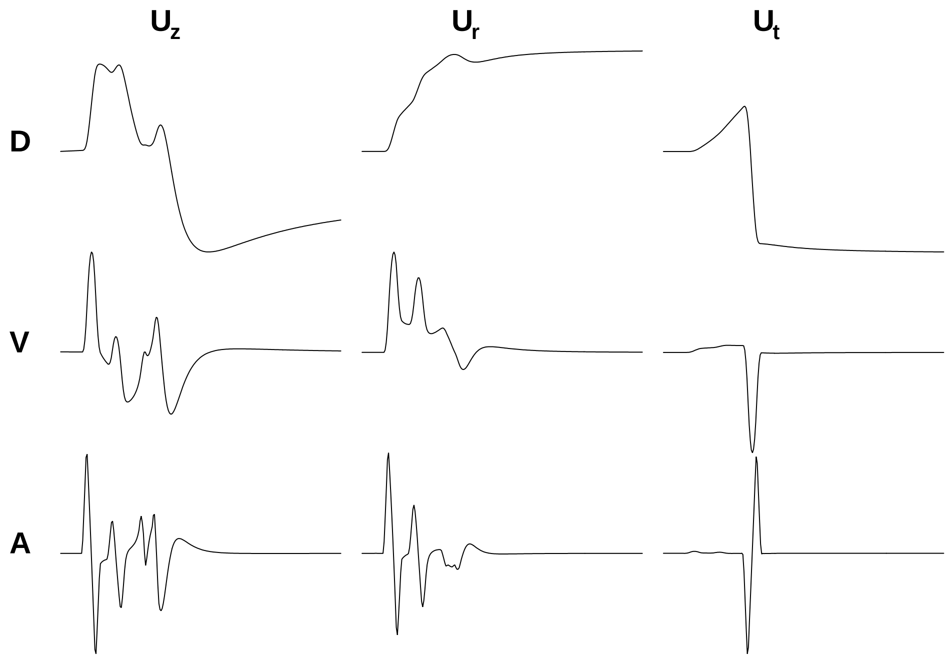

Fig. 5. Halfspace medium response to a point force in the x_1 direction at a location (14.14 14.14 0.0) from a source located at (0,0, 5). In the figure U_z is in the up-direction.

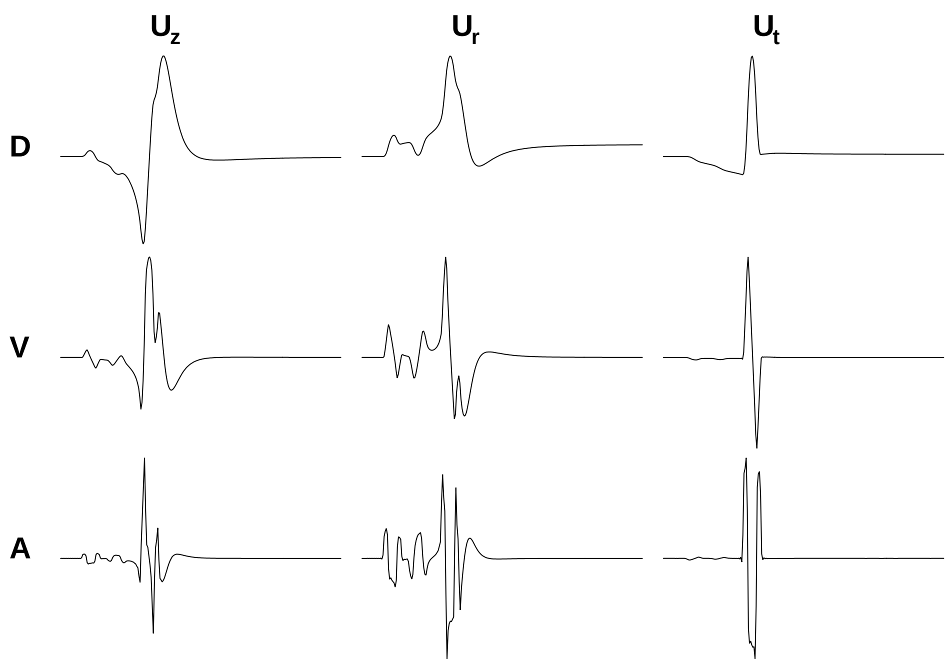

Fig. 6. Halfspace response to a moment tensor force corresponding to a shear disloction with strike 45°, dip 45° and rake 45°. acting at a location (14.14, 14.14, 0.0) from a source located at (0,0, 5). In the figure U_z is in the up-direction.

4. Isotropic Static Displacement

To develop the framework for determining the static deformation in a layered media, we must derive the solution for the wholespace and halfspace first in the wavenumber domain so that we can test the wavenumber integrands derived for the layered medium problem. In addition we also need the solution in the spatial domain to test the numerical wavenumber integration used in the general problem. For a step function in force or moment, we need to consider the limit of the integrands in (5.6.12) as $\omega \rightarrow 0$.

The determination of the limits must be carefully done. The of terms in the integrands seen in (5.6.12) as $\omega \rightarrow 0$ are as follow:

$$\sqrt{k^2 - \omega^2/v^2} \rightarrow k\,(1 - \omega^2/2k^2v^2)$$

$$\frac{1}{\sqrt{k^2 - \omega^2/v^2}} \rightarrow \frac{1}{k}\,(1 + \omega^2/2k^2v^2)$$

$$e^{-\sqrt{k^2 - \omega^2/v^2}\,H} \rightarrow e^{-kH}\,(1 + H\omega^2/2kv^2)$$

4.1 Wholespace solution

After carefully expanding the terms in (5.6.12) and taking the limit, one obtains the following terms where $F_j = \lim_{\omega - 0} F_j(k, \omega)$:

$$F_1 : \ \frac{1}{4\pi\rho}\left[\frac{2}{\alpha^2} - \frac{3}{2}\,k|z|\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)\right] sgn(z) e^{-k|z|}$$

$$F_2 : \ -\frac{1}{4\pi\rho}\left[\frac{2}{\alpha^2} - \frac{3}{2}\,k|z|\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) - \frac{3}{2}\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\right] e^{-k|z|}$$

$$F_3 : \ -\frac{1}{4\pi\rho}\left[\frac{1}{\beta^2} + k|z|\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) - \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\right] e^{-k|z|}$$

$$F_4 : \ \frac{1}{4\pi\rho}\left[k|z|\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) + \frac{1}{\beta^2})\right] e^{-k|z|} sgn(z)$$

$$F_5 : \ -\frac{k}{4\pi\rho}\, e^{-k|z|}\, \frac{z}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)$$

$$F_6 : \ \frac{1}{4\pi\rho}\, e^{-k|z|}\left[\frac{k|z|}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) + \frac{1}{2}\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\right]$$

$$F_7 : \ \frac{1}{4\pi\rho\alpha^2}\, e^{-k|z|} sgn(z)$$

$$F_8 : \ -\frac{1}{4\pi\rho\alpha^2}\, e^{-k|z|}$$

$$F_9:\ -\frac{1}{4\pi\rho k}\, e^{-k|z|}\left[\frac{k|z|}{2}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)-\frac{1}{2}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\right]$$

$$F_{10}:\ \frac{1}{4\pi\rho}\, e^{-k|z|}\,\frac{z}{2}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)$$

$$F_{11}:\ -\frac{1}{4\pi\rho}\, e^{-k|z|}\,\frac{z}{2}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)$$

$$F_{12}:\ \frac{1}{4\pi\rho k}\, e^{-k|z|}\left[\frac{k|z|}{2}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)+\frac{1}{2}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\right]$$

$$F_{13}:\ -\frac{1}{4\pi\rho\beta^2}\, e^{-k|z|} sgn(z)$$

$$F_{14}:\ -\frac{1}{4\pi\rho\beta^2}\, e^{-k|z|}$$

$$F_{15}:\ -\frac{1}{4\pi\rho\beta^2}\, e^{-k|z|}\,\frac{1}{k}$$

$$F_{16}:\ 0$$

When used together with the (5.6.19), the discontinuities in the transformed stress displacement lead to the following source representation givenin Table 5.X.

Table 5.X/Source Term Coefficients - Isotropic Medium

Term	ΔU_r	ΔU_z	ΔT_z	ΔT_r	ΔU_ϕ	ΔT_ϕ	n
DD	0	$\frac{1}{\pi(\lambda+2\mu)}$	0	$-\frac{k}{2\pi}\,\frac{3\lambda+2\mu}{\lambda+2\mu}$	0	0	0
DS	$\frac{1}{2\pi\mu}$	0	0	0	$-\frac{1}{2\pi\mu}$	0	1
SS	0	0	0	$-\frac{k}{2\pi}$	0	$\frac{k}{2\pi}$	2
EX	0	$\frac{1}{2\pi(\lambda+2\mu)}$	0	$\frac{k}{\pi}\,\frac{\mu}{\lambda+2\mu}$	0	0	0
VF	0	0	$-\frac{1}{2\pi}$	0	0	0	0
HF	0	0	0	$-\frac{1}{2\pi}$	0	$\frac{1}{2\pi}$	1

Here we use $\rho\alpha^2 = \lambda+2\mu$ and $\rho\beta^2 = \mu$.

After placing these expressions into (5.6.10) and evaluating the wavenumebr integrals using the expressions in Section 2 of the Appendix, we obtain the following algebraic expressions for the static deformation in a wholespace.

$$ZDS = \frac{1}{4\pi\rho}\left[\frac{1}{\beta^2}\frac{r}{R^3} + \left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)\frac{3rz^2}{R^5} - \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\frac{r}{R^3}\right]$$

$$RDS = \frac{1}{4\pi\rho}\left[\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)\left(\frac{3z^3}{R^5} - \frac{2z}{R^3}\right) + \frac{z}{R^3}\right]$$

$$TDS = -\frac{1}{4\pi\rho}\frac{1}{\alpha^2}\frac{z}{R^3}$$

$$ZSS = -\frac{1}{4\pi\rho}\frac{1}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)\frac{3zr^2}{R^5}$$

$$RSS = \frac{1}{4\pi\rho}\frac{r}{R^3}\frac{1}{2}\left[\left(\frac{3z^2}{R^2} - 2\right)\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\right]$$

$$TSS = -\frac{1}{4\pi\rho}\frac{1}{\alpha^2}\frac{r}{R^3}$$

$$ZEX = \frac{1}{4\pi\rho}\frac{1}{\alpha^2}\frac{z}{R^3}$$

$$REX = \frac{1}{4\pi\rho}\frac{1}{\alpha^2}\frac{r}{R^3}$$

$$ZVF = \frac{1}{4\pi\rho}\frac{1}{2}\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\frac{1}{R} - \frac{1}{4\pi\rho}\frac{1}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)\frac{z^2}{R^3}$$

$$RVF = -\frac{1}{4\pi\rho}\frac{1}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)\frac{rz}{R^3}$$

$$ZHF = -\frac{1}{4\pi\rho}\frac{1}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)\frac{rz}{R^3}$$

$$RHF = \frac{1}{4\pi\rho}\frac{1}{2}\left[-\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)\frac{r^2}{R^3} + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\frac{1}{R}\right]$$

$$THF = -\frac{1}{4\pi\rho}\frac{1}{2}\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\frac{1}{R}$$

4.2 Halfspace solution

The required low-frequency limit of the surface expressions give the following terms for the surface deformation due to a buried source with a step-like source time function.

$$F_1 = \frac{\frac{\beta^2}{\alpha^2} - 3kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2\left(\frac{\beta^2}{\alpha^2} - 1\right)} e^{-kh}$$

$$F_2 = -\frac{\left(3 - 4\frac{\beta^2}{\alpha^2}\right) + 3kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2\left(\frac{\beta^2}{\alpha^2} - 1\right)} e^{-kh}$$

$$F_3 = \frac{2kh}{4\pi\rho\beta^2\left(\frac{\beta^2}{\alpha^2} - 1\right)} e^{-kh}$$

$$F_4 = -\frac{2}{4\pi\rho\beta^2}(1 - kh)\, e^{-kh}$$

$$F_5 = -\frac{\frac{\beta^2}{\alpha^2} + kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2\left(\frac{\beta^2}{\alpha^2} - 1\right)} e^{-kh}$$

$$F_6 = -\frac{1 + kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2\left(\frac{\beta^2}{\alpha^2} - 1\right)} e^{-kh}$$

$$F_7 = \frac{2}{4\pi\rho(\beta^2 - \alpha^2)} e^{-kh}$$

$$F_8 = \frac{2}{4\pi\rho(\beta^2 - \alpha^2)} e^{-kh}$$

$$F_9 = -\frac{1 - kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2 k\left(\frac{\beta^2}{\alpha^2} - 1\right)} e^{-kh}$$

$$F_{10} = \frac{-\frac{\beta^2}{\alpha^2} + kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2 k\left(\frac{\beta^2}{\alpha^2} - 1\right)} e^{-kh}$$

$$F_{11} = -\frac{\frac{\beta^2}{\alpha^2} + kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2 k\left(\frac{\beta^2}{\alpha^2} - 1\right)}\, e^{-kh}$$

$$F_{12} = -\frac{1 + kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2 k\left(\frac{\beta^2}{\alpha^2} - 1\right)}\, e^{-kh}$$

$$F_{13} = \frac{2}{4\pi\rho\beta^2}\, e^{-kh}$$

$$F_{14} = -\frac{2}{4\pi\rho\beta^2}\, e^{-kh}$$

$$F_{15} = -\frac{2}{4\pi\rho\beta^2 k}\, e^{-kh}$$

$$\lim_{\omega\to 0} \omega^2 F_R = 2k^2\beta^2\left(\frac{\beta^2}{\alpha^2} - 1\right)$$

The expressions for F_1 through F_8 agree with those of Herrmann and Wang (1985) once one accounts for differences in the direction of positive vertical displacement and the form of integration for the vertical component. The contributions leading to the point force response given by F_9, F_{10}, F_{11}, F_{12}, are a new contribution.

We do not present closed form solutions for the ZDD, etc., but rather leave it to the reader to place these expressions into (5.6.10) and then to use the Bessel integral expressions given in A.2. We do note that at the top of the halfspace, $TDS \equiv 0$.

5. Isotropic equations of motion, eigenvalues and eigenvectors

Following Zhu and Rivera (2002), we will formulate the static problem in layered media first for isotropic media and then transverse isotropic media. We will start with the homogeneous equations of motion (5.6.20) as $\omega \to 0$.

5.1 SH Problem:

The system of differential equations to be solved is

$$\frac{d}{dz}\begin{bmatrix} U_\phi \\ T_\phi \end{bmatrix} = \begin{bmatrix} 0 & 1/\mu \\ \mu k^2 & 0 \end{bmatrix}\begin{bmatrix} U_\phi \\ T_\phi \end{bmatrix}$$

The displacement-stress vector **B(z)** is given by

$$\mathbf{B} = \mathbf{E}\Lambda\mathbf{K}$$

where

$$\mathbf{E} = \begin{bmatrix} k & k \\ \mu k^2 & -\mu k^2 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} e^{kz} & 0 \\ 0 & e^{-kz} \end{bmatrix}$$

and

$$\mathbf{K} = \begin{bmatrix} K^U \\ K_D \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} e^{kz} & 0 \\ 0 & e^{-kz} \end{bmatrix}$$

From these we obtain

$$\mathbf{E}^{-1} = \frac{1}{2} \begin{bmatrix} \frac{1}{k} & \frac{1}{k} \\ \frac{1}{\mu k^2} & -\frac{1}{\mu k^2} \end{bmatrix}$$

and the propagator matrix relating the displacement-stress vectors from the top to the bottom of a layer with thickness z is

$$\mathbf{A} = \mathbf{E}\Lambda\mathbf{E}^{-1} = \begin{bmatrix} \cosh kz & \sinh kz/\mu k \\ \mu k \sinh kz & \cosh kz \end{bmatrix}$$

which we recognize as the $\lim_{\omega \to 0}$ of (7.3.3)

5.2 P-SV Problem:

The system of differential equations to be solved is

$$\frac{d}{dz} \begin{bmatrix} U_r \\ U_z \\ T_z \\ T_r \end{bmatrix} = \begin{bmatrix} 0 & -k & 0 & \frac{1}{\mu} \\ \frac{k\lambda}{\lambda + 2\mu} & 0 & \frac{1}{\lambda + 2\mu} & 0 \\ 0 & 0 & 0 & k \\ \frac{4k^2 \mu(\lambda + \mu)}{\lambda + 2\mu} & 0 & \frac{-k\lambda}{\lambda + 2\mu} & 0 \end{bmatrix} \begin{bmatrix} U_r \\ U_z \\ T_z \\ T_r \end{bmatrix}$$

The decomposition $\mathbf{B} = \mathbf{E}\Lambda\mathbf{K}$ is complicated since the eigenvalues of this matrix are k, k, $-k$ and $-k$ and the λ matrix is no longer diagonal. Following the development of Zhu and Rivera (2002) the decomposition for the system above is

$$\lambda = \begin{bmatrix} e^{kz} & kze^{kz} & 0 & 0 \\ 0 & e^{kz} & 0 & 0 \\ 0 & 0 & e^{-kz} & kze^{-kz} \\ 0 & 0 & 0 & e^{-kz} \end{bmatrix}$$

and

$$\mathbf{E}=\begin{bmatrix} k & \dfrac{k\mu}{\lambda+\mu} & k & -\dfrac{k\mu}{\lambda+\mu} \\ k & -k\,\dfrac{\lambda+2\mu}{\lambda+\mu} & -k & -k\,\dfrac{\lambda+2\mu}{\lambda+\mu} \\ 2\mu k^2 & -2\mu k^2 & 2\mu k^2 & 2\mu k^2 \\ 2\mu k^2 & 0 & -2\mu k^2 & 0 \end{bmatrix}$$

Note that since the eigenvector matrix Λ is not diagonal, the second column of **E** is related to the first column, so that one must not multiply one of the columns by a scalar without applying the same multiplication to the other column. This also applies to the third and fourth columns. Note also that this differs from Zhu and Rivera (2002) in that all occurrences of z here appear in combination with k, e.g., as kz to preserve the dimension of the matrix.

The required inverse of the eigenvector matrix is

$$\mathbf{E}^{-1}=\frac{\lambda+\mu}{2k(\lambda+2\mu)}\begin{bmatrix} 1 & 0 & \dfrac{1}{2k(\lambda+\mu)} & \dfrac{\lambda+2\mu}{2\mu k(\lambda+\mu)} \\ 1 & -1 & -\dfrac{1}{2\mu k} & \dfrac{1}{2\mu k} \\ 1 & 0 & \dfrac{1}{2k(\lambda+\mu)} & -\dfrac{\lambda+2\mu}{2\mu k(\lambda+\mu)} \\ -1 & -1 & \dfrac{1}{2\mu k} & \dfrac{1}{2\mu k} \end{bmatrix}$$

The elements of the propagator matrix $\mathbf{E}\Lambda\mathbf{E}^{-1}$ are:

$$a_{11}=C+\frac{\lambda+\mu}{\lambda+2\mu}\,S\,kz$$

$$a_{12}=-\left(\frac{\lambda+\mu}{\lambda+2\mu}\,C\,kz+\frac{\mu}{\lambda+2\mu}\,S\right)$$

$$a_{13}=-\frac{\lambda+\mu}{\lambda+2\mu}\,\frac{1}{2\mu k}\,S\,kz$$

$$a_{14}=\frac{1}{2\mu k}\left(\frac{\lambda+3\mu}{\lambda+2\mu}\,S+\frac{\lambda+\mu}{\lambda+2\mu}\,C\,kz\right)$$

$$a_{21}=\frac{1}{\lambda+2\mu}\,(-\mu S+(\lambda+\mu)C\,kz)$$

$$a_{22}=C-\frac{\lambda+\mu}{\lambda+2\mu}\,S\,kz$$

$$a_{23}=\frac{\lambda+\mu}{\lambda+2\mu}\,\frac{1}{2\mu k}\left(\frac{\lambda+3\mu}{\lambda+\mu}\,S-C\,kz\right)$$

$$a_{24}=-\,a_{13}$$

$$a_{31}=\frac{\lambda+\mu}{\lambda+2\mu}\,2\mu kS\,kz$$

$$a_{32}=\frac{\lambda+\mu}{\lambda+2\mu}\,2\mu k(S-C\,kz)$$

$$a_{33}=a_{22}$$

$$a_{34}=-\,a_{12}$$

$$a_{41}=\frac{\lambda+\mu}{\lambda+2\mu}\,2\mu k(S+C\,kz)$$

$$a_{42}=-\,a_{31}$$

$$a_{43}=-\,a_{21}$$

$$a_{44}=a_{11}$$

where here $C=\cosh kz$, $S=\sinh kz$. Note that the propagator matrix elements can also be obtained by applying $\lim_{\omega \to 0}$ to the expression given in §7.15 (Zhu and Rivera, 2002).

For completeness, the compound matrix elements are defined as $A_{IJ} = A|_{ij}^{kl} \equiv a_{ik}a_{jl} - a_{il}a_{jk}$ with $ij = (12, 13, 14, 23, 24, 34)$ for $I = (1, 2, 3, 4)$, respectively. The kl are similarly defined in terms of J. The elements of the compound matrix A are just sub-determinants of the a matrix. The terms are as follow:

$$A_{11} = A_{22} + (1 - A_{33})$$

$$A_{12} = \frac{\lambda + \mu}{\lambda + 2\mu} \frac{1}{2\mu k} \left[\frac{\lambda + 3\mu}{\lambda + \mu} SC - kz \right]$$

$$A_{13} = \frac{1}{\lambda + 2\mu} \frac{1}{\lambda + 2\mu} \frac{1}{2\mu k} \left[\mu(\lambda + 3\mu)S^2 - (\lambda + \mu)^2 k^2 z^2 \right]$$

$$A_{14} = - A_{13}$$

$$A_{15} = - \frac{1}{\lambda + 2\mu} \frac{1}{2\mu k} \left[(\lambda + 3\mu)SC - (\lambda + \mu)kz \right]$$

$$A_{16} = \frac{1}{(\lambda + 2\mu)^2} \frac{1}{(2\mu k)^2} \left[-(\lambda + 3\mu)^2 S^2 + (\lambda + \mu)^2 k^2 z^2 \right]$$

$$A_{21} = 2\mu k \frac{\lambda + \mu}{\lambda + 2\mu} (SC - kz)$$

$$A_{22} = C^2$$

$$A_{23} = \frac{1}{\lambda + 2\mu} \left[\mu SC + (\lambda + \mu)kz \right]$$

$$A_{24} = - A_{23}$$

$$A_{25} = - S^2$$

$$A_{26} = A_{15}$$

$$A_{31} = 2\mu k \frac{\lambda + \mu}{\lambda + 2\mu} \left[\frac{\mu}{\lambda + 2\mu} S^2 + \frac{\lambda + \mu}{\lambda + 2\mu} k^2 z^2 \right]$$

$$A_{32} = \frac{1}{\lambda + 2\mu} \left[\mu CS - (\lambda + \mu)kz \right]$$

$$A_{33} = 1 + \left(\frac{\mu}{\lambda + 2\mu} \right)^2 S^2 - \left(\frac{\lambda + \mu}{\lambda + 2\mu} \right)^2 k^2 z^2$$

$$A_{34} = 1 - A_{33}$$

$$A_{35} = -A_{23}$$

$$A_{36} = -A_{13}$$

$$A_{41} = -A_{31}$$

$$A_{42} = -A_{32}$$

$$A_{43} = 1 - A_{44}$$

$$A_{44} = A_{33}$$

$$A_{45} = A_{23}$$

$$A_{46} = A_{13}$$

$$A_{51} = -2\mu k \frac{\lambda + \mu}{\lambda + 2\mu} (SC + kz)$$

$$A_{52} = -S^2$$

$$A_{53} = -A_{32}$$

$$A_{54} = -A_{53} = A_{32}$$

$$A_{55} = A_{22}$$

$$A_{56} = A_{12}$$

$$A_{61} = -\left(2\mu k \frac{\lambda + \mu}{\lambda + 2\mu}\right)^2 \left(S^2 - k^2 z^2\right)$$

$$A_{62} = A_{51}$$

$$A_{63} = -A_{31}$$

$$A_{64} = -A_{63} = A_{31}$$

$$A_{65} = A_{21}$$

$$A_{66} = A_{11}$$

There are a number of important features of this matrix:

$$A_{3j} = -A_{4j}\ , \quad j \neq 3, 4$$

$$A_{i3} = -A_{i4}\ , \quad j \neq 3, 4$$

$$A_{33} = A_{44} ,$$

$$A_{34} = A_{43} = 1 - A_{44} = 1 - A_{33}$$

In §7.4, we noted that the boundary condition for a lower halfspace requires the matrix $\mathbf{G} = \mathbf{E}^{-1}$ and that for the upper halfspace requires $\mathbf{H} = \mathbf{E}$. When using the compund matrix formulation, we require the following elements:

$$G|_{12}^{12} = -\left[\frac{\lambda + \mu}{2k(\lambda + 2\mu)} \right]^2 \qquad G|_{13}^{12} = -\left[\frac{\lambda + \mu}{2k(\lambda + 2\mu)} \right]^2 \frac{\lambda + 2\mu}{2k\mu(\lambda + \mu)}$$

$$G|_{14}^{12} = -\left[\frac{\lambda + \mu}{2k(\lambda + \mu)} \right]^2 \frac{1}{2k(\lambda + \mu)} \qquad G|_{23}^{12} = \left[\frac{\lambda + \mu}{2k(\lambda + 2\mu)} \right]^2 \frac{1}{2k(\lambda + \mu)}$$

$$G|_{24}^{12} = \left[\frac{\lambda + \mu}{2k(\lambda + 2\mu)} \right]^2 \frac{\lambda + 2\mu}{2\mu k(\lambda + \mu)} \qquad G|_{34}^{12} = \left[\frac{\lambda + \mu}{2k(\lambda + 2\mu)} \right]^2 \frac{\lambda + 3\mu}{4k^2 \mu^2 (\lambda + \mu)}$$

$$H|_{12}^{12} = -k^2 \frac{\lambda + 3\mu}{\lambda + \mu} \qquad H_{12}^{13} = -2\mu k^3 \frac{\lambda + 2\mu}{\lambda + \mu}$$

$$H|_{12}^{14} = -2k^3 \frac{\mu^2}{\lambda + \mu} \qquad H_{12}^{23} = 2k^3 \frac{\mu^2}{\lambda + \mu}$$

$$H|_{12}^{24} = 2\mu k^3 \frac{\lambda + 2\mu}{\lambda + \mu} \qquad H_{12}^{34} = 4\mu^2 k^4$$

The use of the matrices in the computation of the medium response is discussed in Chapter 7.

Wavenumber integration

The evaluation of the wavenumber integrals requires care. Numerical evaluation introduces error through the finite integration limits and through the particular integration rule used. Items of concern are the infinite limit of integration in wavenumber, the sample interval and the integration scheme used.

Infinite Limits

The problem is to approximate the infinite integral by a finite one. The obvious approach is to truncate the integral so that

$$\int_0^{\infty} f(k,r)dk \approx \int_0^{k_{\max}} f(k,r)dk\,,$$

but now the choice of $k_{\max}$ becomes important, especially when the function $f(k,r)$ is significantly different from zero for $k > k_{\max}$.

To illustrate this, consider the *REX* function for an elastic wholespace:

$$REX = \int_0^{\infty} \frac{1}{4\pi\rho\alpha^2} e^{-k|z|} J_1(kr)kdk$$

$$= -\frac{1}{4\pi\rho\alpha^2} \frac{\partial F}{\partial r}$$

$$= \frac{1}{4\pi\rho\alpha^2} \left(\frac{r}{R^3} \right)$$

$$= \int_0^{k_{\max}} \frac{1}{4\pi\rho\alpha^2} e^{-k|z|} J_1(kr)kdk + Error$$

where $R^2 = r^2 + z^2$.

Since this integral must be evaluated numerically, there are several problems to be faced. The most obvious is the choice of $k_{\max}$. If one makes $k_{\max}|z|$ large, then the error due to the integration in the range $[k_{\max}, \infty]$ is presumably small because the integrand is negligible. A problem arises when $|z|$ is small. The required $k_{\max}$ will be large and the numerical integration will involve more steps, each of which can contribute a rounding error.

To control the error due to the choice of $k_{\max}$, assume that a function $g(k,r)$ exists such that $g(k,r) \approx f(k,r)$ as $k \to \infty$, and that $\int_0^{\infty} g(k,r)dk = G(r)$. Thus

$$\int_0^{\infty} f(k,r)dk = \int_0^{\infty} \left[f(k,r) - g(k,r) \right] dk + G(r)$$

$$\approx \int_0^{k_{max}} \left[f(k,r) - g(k,r) \right] dk + G(r)$$

where k_{max} is such that now the integral

$$\int_{k_{max}}^{\infty} \left[f(k,r) - g(k,r) \right] dk$$

is negligible. Since the integrand here $\left[f(k,r) - g(k,r) \right]$ is designed to be less than the original integrand $f(k,r)$ we have ensured that the truncation error introduced by the use of k_{max} will be much smaller.

The next step is to approximate the integral numerically. The current approach is to use a simple rectangular integration rule:

$$\int_0^{\infty} f(k,r)dk \approx \sum_{i=NK}^{1} \left[f(k_i,r) - g(k_i,r) \right] \Delta k + G(r)$$

where $k_i = (i - \frac{1}{2})\Delta k$. The order of the summation loop is expressed this way to reduce roundoff error by adding small terms first. The current procedure defines NK, Δk and three wavenumbers k_1, k_2 and k_3 as follows:

$|z| < r$

$r > 0$

$\Delta k = (1/8) 2\pi/r$
$k_{max} = 9/|z|$
$NK = k_{max}/\Delta k$

$r \equiv 0$

$k_{max} = 9/|z|$
$NK = 100$
$\Delta k = K_{max}/NK$

$k_1 = 3/|z|$
$k_2 = 6/|z|$
$k_3 = 8/|z|$

$|z| \geq r$

$\Delta k = 64 \cdot 2\pi/r$
$k_{max} = 9 \cdot 2\pi/r$
$NK = k_{max}/\Delta k$
$k_1 = 0.3 k_{max}$
$k_2 = 0.6 k_{max}$
$k_3 = 0.8 k_{max}$

To illustrate the way that an asymptotic solution of formed, consider the integral

$$\int_0^{\infty} F(k) J_n(kr) dk$$

and

$$\int_0^\infty (A + Bk + Ck^2) e^{-k|h|} J_n(kr)\, dk$$

The latter has an analytic solution given in A.2, which we define G(r). Thus the first integral is now written as

$$\int_0^\infty F(k) J_n(kr)\, dk = \int_0^\infty [F(k) - (A + Bk + Ck^2) e^{-k|h|}] J_n(kr)\, dk + G(r)$$

The polynomial coefficients are determined by solving the equations

$$(A + Bk_j + Ck_j^2) e^{-k_j|h|} = F(k_j) \quad \text{for } j = 1, 2, 3$$

There is nothing special about using a rectangular rule. The implementation of an adaptive method of integration may be worth investigating.

References:

Harkrider, D. G., and D. V. Helmberger (1978). A note on nonequivalent quadrupole source cylindrical shear potentials which give equal displacements, *Bull. Seism. Soc. Am.* **68,** 125-132.

Sato, R., and M. Matsu'ura (1973). Static deformations due to the fault spreading over several layers in a multi-layered medium. Part I: Displacement, *J. Phys. Earth (Tokyo)* **21,** 227-249

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Appendix

A.1. Moment tensor for a shear dislocation

Consider the fault plane show in Figure A.1, whose parameters are defined by the strike, dip and rake angles.

For a shear dislocation in an isotropic medium, the moment tensor elements are

$$M_{11} = -M_0(\sin\delta \cos\lambda \sin 2\phi_f + \sin 2\delta \sin\lambda \sin^2\phi_f)$$

$$M_{12} = M_0(\sin\delta \cos\lambda \cos 2\phi_f + \tfrac{1}{2} \sin 2\delta \sin\lambda \sin 2\phi_f)$$

$$M_{13} = -M_0(\cos\delta \cos\lambda \cos\phi_f + \cos 2\delta \sin\lambda \sin\phi_f)$$

$$M_{22} = M_0(\sin\delta \cos\lambda \sin 2\phi_f - \sin 2\delta \sin\lambda \cos^2\phi_f)$$

$$M_{23} = -M_0(\cos\delta \cos\lambda \sin\phi_f - \cos 2\delta \sin\lambda \cos\phi_f)$$

$$M_{33} = M_0 \sin 2\delta \sin\lambda$$

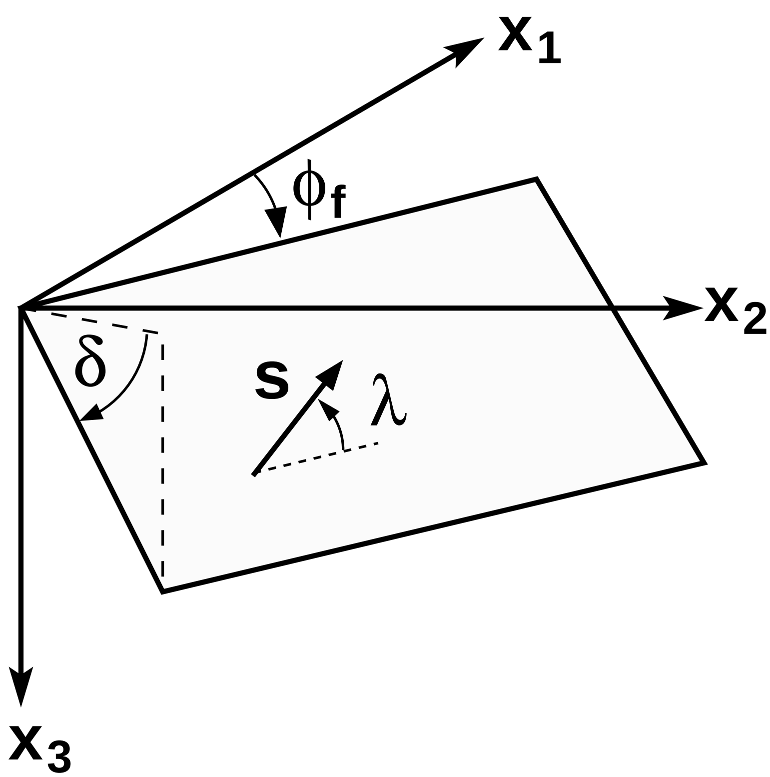

Fig. A.1. Fault plane angle convention. The x_1, x_2 and x_3 axes are in the north, east and down directions. *phif* is the strike, measured north, δ is the dip, measured downward from a horizontal direction perpendicular to the strike, and λ is the rake angle indicating the direction of motion on the fault, given by the vector **s**. The side of the fault nearest the viewer will move in the **s** direction.

$$M_{31} = M_{13}$$

$$M_{32} = M_{23}$$

$$M_{21} = M_{12}$$

A.2. Bessel function integrals

To convert the expressions for the static solution in the wavenumber domain to the sparial domain, we use the following integrals of the Bessel functions:

$$F = \int_0^\infty e^{-k|z|} J_0(kr)\,dk = \frac{1}{R}$$

$$-\frac{\partial F}{\partial z} = sgn(z) \int_0^\infty k e^{-k|z|} J_0(kr)\,dk = \frac{z}{R^3}$$

$$-\frac{\partial F}{\partial r} = \int_0^\infty k e^{-k|z|} J_1(kr)\,dk = \frac{r}{R^3}$$

$$\frac{\partial^2 F}{\partial r \partial z} = sgn(z) \int_0^\infty k^2 e^{-k|z|} J_1(kr) dk = \frac{3rz}{R^5}$$

$$\frac{\partial^2 F}{\partial z^2} = \int_0^\infty k^2 e^{-k|z|} J_0(kr) dk = \left(\frac{3z^2}{R^5} - \frac{1}{R^3} \right)$$

$$\frac{\partial^2 F}{\partial r^2} = -\int_0^\infty k e^{-k|z|} \left(kJ_0(kr) - \frac{1}{r} J_1(kr) \right) dk = \left(\frac{3r^2}{R^5} - \frac{1}{R^3} \right)$$

$$\frac{\partial^3 F}{\partial r^3} = \int_0^\infty k^2 e^{-k|z|} \left(kJ_1(kr) - \frac{1}{r} J_2(kr) \right) dk = -\left(\frac{15r^3}{R^7} - \frac{9r}{R^5} \right)$$

$$\frac{\partial^3 F}{\partial z^3} = - sgn(z) \int_0^\infty k^3 e^{-k|z|} J_0(kr)\, dk = -\left(\frac{15z^3}{R^7} - \frac{9z}{R^5} \right)$$

$$\frac{\partial F}{\partial r^2 \partial z} = sgn(z) \int_0^\infty k^2 e^{-k|z|} \left(kJ_0(kr) - \frac{1}{r} J_1(kr) \right) dk = -\frac{15r^2 z}{R^7} - \frac{3z}{R^5}$$

$$\frac{\partial F}{\partial r \partial z^2} = -\int_0^\infty k^3 e^{-k|z|} J_1(kr)\, dk = -\frac{15rz^2}{R^7} - \frac{3r}{R^5}$$

Note that these reltions also follow from §5.9 in the limit as $i\omega \rightarrow 0$.

The following integrals are taken from Harkrider and Helmberger (1978):

$$\int_0^\infty e^{-k|z|} k^{-1} J_1(kr) dk = \frac{R - |z|}{r}$$

$$\int_0^\infty e^{-k|z|} J_2(kr) dk = \frac{1}{R} \frac{(R - |z|)^2}{r^2}$$

$$\int_0^\infty e^{-k|z|} J_n(kr) dk = \frac{1}{R} \left(\frac{R - |z|}{r} \right)^n$$

We can summarize these results in a table. For example, factoring the $sgn(z)$ in the second equation leads to

$$\int_0^\infty k e^{-k|z|} J_0(kr) dk = \frac{|z|}{R^3}$$

For the static problem, the wavenumber integrals are of the form $JnKm \equiv \int_0^\infty e^{-k|z|} J_n(kr) k^m\, dk$ and are explicitly given in Table 1.

Table 1. Wavenumber integrals

n	m	$JnKm$	n	m	$JnKm$
0	0	$\frac{1}{R}$	2	0	$\frac{R}{r^2}\left(1-2\frac{\lvert z\rvert}{R}+\frac{z^2}{R^2}\right)$
0	1	$\frac{\lvert z\rvert}{R^3}$	2	1	$\frac{1}{r^2}\left(2-3\frac{\lvert z\rvert}{R}+\frac{\lvert z\rvert^3}{R^3}\right)$
0	2	$\frac{(2z^2-r^2)}{R^5}$	2	2	$\frac{3r^2}{R^5}$
1	0	$\frac{r(R+\lvert z\rvert)}{R}$			
1	1	$\frac{r}{R^3}$			
1	2	$\frac{3r\lvert z\rvert}{R^5}$			

where $R=\sqrt{r^2+z^2}$