1 Introduction

This note reviews expressions for the cylindrical coordinate solution of elastic wave propagation for models in which material properties depend only the the z coordinate. The Fourier transform of the solution is developed in terms of wavenumber integrals of Bessel functions, e.g., the Fourier-Bessel transforms. This overview will set the stage for understanding the numerical problems associated with approximating these transforms.

For isotropic and transverse isotropic media in which the material properties only vary in the z-direction, the form of the displacements in cylindrical coordinates can be expressed as

$$\begin{aligned}
u_z(r,z,\omega) &= \sum_n \left(A_n \cos n\phi + B_n \sin n\phi \right) \\
&\quad \cdot \int_0^\infty U_z(k,z,\omega) J_n(kr) k dk \\
u_r(r,z,\omega) &= \sum_n \left(A_n \cos n\phi + B_n \sin n\phi \right) \\
&\quad \cdot \int_0^\infty \left[U_r(k,z,\omega) \frac{\partial J_n(kr)}{\partial r} \quad - \frac{n}{r} U_\phi(k,z,\omega) J_n(kr) \right] dk \\
u_\phi(r,z,\omega) &= \sum_n \left(A_n \sin n\phi - B_n \cos n\phi \right) \\
&\quad \cdot \int_0^\infty \left[U_\phi(k,z,\omega) \frac{\partial J_n(kr)}{\partial r} \quad - \frac{n}{r} U_r(k,z,\omega) J_n(kr) \right] dk.
\end{aligned}$$

The U_r, U_ϕ and U_z are solutions are a differential equation in z. This explicitly shows the ϕ and r dependence.

The transformed stresses in isotropic media are defined as

$$\begin{aligned}
T_r &= \mu \left(\frac{dU_r}{dz} + kU_z \right) \\
T_z &= (\lambda + 2\mu) \frac{dU_z}{dz} - k\lambda U_r \\
T_\phi &= \mu \frac{dU_\phi}{dz}
\end{aligned}$$

The transformed displacement and stress functions can be obtained using propagator or reflection matrices.

The Fourier transform of cylindrical coordinate displacements due to point force and moment tensor sources can be written in terms of basic Green's functions as follows:

$$\begin{aligned}
u_z(r,z,h,\omega) &= (F_1\cos\phi \ + \ F_2\sin\phi)\,ZHF \ + \ F_3 ZVF \\
&+ \ M_{11}\left[\frac{ZSS}{2}\ \cos(2\phi) \ - \ \frac{ZDD}{6} \ + \ \frac{ZEX}{3}\right] \\
&+ \ M_{22}\left[\frac{-ZSS}{2}\ \cos(2\phi) \ - \ \frac{ZDD}{6} \ + \ \frac{ZEX}{3}\right] \\
&+ \ M_{33}\left[\frac{ZDD}{3} \ + \ \frac{ZEX}{3}\right] \\
&+ \ M_{12}\left[ZSS\ \sin(2\phi)\right] \\
&+ \ M_{13}\left[ZDS\ \cos(\phi)\right] \\
&+ \ M_{23}\left[ZDS\ \sin(\phi)\right] \\
u_r(r,z,h,\omega) &= (F_1\cos\phi \ + \ F_2\sin\phi)\,RHF \ + \ F_3 RVF \\
&+ \ M_{11}\left[\frac{RSS}{2}\ \cos(2\phi) \ - \ \frac{RDD}{6} \ + \ \frac{REX}{3}\right] \\
&+ \ M_{22}\left[\frac{-RSS}{2}\ \cos(2\phi) \ - \ \frac{RDD}{6} \ + \ \frac{REX}{3}\right] \\
&+ \ M_{33}\left[\frac{RDD}{3} \ + \ \frac{REX}{3}\right] \\
&+ \ M_{12}\left[RSS\ \sin(2\phi)\right] \\
&+ \ M_{13}\left[RDS\ \cos(\phi)\right] \\
&+ \ M_{23}\left[RDS\ \sin(\phi)\right] \\
u_\phi(r,z,h,\omega) &= (F_1\sin\phi - F_2\cos\phi)\,THF \\
&+ \ M_{11}\left[\frac{TSS}{2}\ \sin(2\phi)\right] \\
&+ \ M_{22}\left[\frac{-TSS}{2}\ \sin(2\phi)\right] \\
&+ \ M_{12}\left[-TSS\ \cos(2\phi)\right] \\
&+ \ M_{13}\left[TDS\ \sin(\phi)\right] \\
&+ \ M_{23}\left[-TDS\ \cos(\phi)\right]\ .
\end{aligned}$$

Here r is the epicentral distance, h is the source depth, z is the receiver depth, ϕ is the azimuth with respect to the source, and ω is the angular frequency. The F_j and M_{ij} are the Fourier transforms of the point force and point moment tensor source time functions.

The strains expressed in terms of the cylindrical coordinate system displacements are

$$\begin{array}{lll}
e_{rr} = \dfrac{\partial u_r}{\partial r} & e_{r\phi} = \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \phi} + \dfrac{\partial u_\phi}{\partial r} - \dfrac{u_\phi}{r}\right) & e_{\phi\phi} = \dfrac{1}{r}\left(\dfrac{\partial u_\phi}{\partial \phi} + u_r\right) \\
e_{rz} = \dfrac{1}{2}\left(\dfrac{\partial u_z}{\partial r} + \dfrac{\partial u_r}{\partial z}\right) & e_{zz} = \dfrac{\partial u_z}{\partial z} & e_{\phi z} = \dfrac{1}{2}\left(\dfrac{\partial u_\phi}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \phi}\right)
\end{array}$$

with the dilatation Δ is given by

$$\Delta \ = \ \nabla\cdot\mathbf{u} \ = \ \frac{1}{r}\frac{\partial(ru_r)}{\partial r} \ + \ \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} \ + \ \frac{\partial u_z}{\partial z} = e_{rr} + e_{\phi\phi} + e_{zz}.$$

To compute the strains, it is necessary to obtain the partial derivatives of the displacements with respect to r, z and ϕ. The last partial is easily obtained by taking derivatives of the $\sin\phi$ and $\cos\phi$ functions. The partial with respect to z is obtained from the definition of the transformed stress, and the partial with respect to r is obtained using the properties of the Bessel functions.

Except for the cases of a uniform, isotropic wholespace and halfspace, there are no explicit expressions for the Fourier transformed displacements.

2 Wavenumber integration

The specific integrals for the Green's functions used above are

$$
\begin{aligned}
ZDD &= \int_0^\infty F_1(k,\omega)J_0(kr)kdk \\
RDD &= -\int_0^\infty F_2(k,\omega)J_1(kr)kdk \\
ZDS &= \int_0^\infty F_3(k,\omega)J_1(kr)kdk \\
RDS &= \int_0^\infty F_4(k,\omega)J_0(kr)kdk \\
&\quad - \frac{1}{r}\int_0^\infty \left[F_4(k,\omega) \;+\; F_{13}(k,\omega)\right] J_1(kr)dk \\
TDS &= \int_0^\infty F_{13}(k,\omega)J_0(kr)kdk \\
&\quad - \frac{1}{r}\int_0^\infty \left[F_4(k,\omega) \;+\; F_{13}(k,\omega)\right] J_1(kr)dk \\
ZSS &= \int_0^\infty F_5(k,\omega)J_2(kr)kdk \\
RSS &= \int_0^\infty F_6(k,\omega)J_1(kr)kdk \\
&\quad - \frac{2}{r}\int_0^\infty \left[F_6(k,\omega) \;+\; F_{14}(k,\omega)\right] J_2(kr)dk \\
TSS &= \int_0^\infty F_{14}(k,\omega)J_1(kr)kdk \\
&\quad - \frac{2}{r}\int_0^\infty \left[F_6(k,\omega) \;+\; F_{14}(k,\omega)\right] J_2(kr)dk \\
ZEX &= \int_0^\infty F_7(k,\omega)J_0(kr)kdk \\
REX &= -\int_0^\infty F_8(k,\omega)J_1(kr)kdk \\
ZVF &= \int_0^\infty F_9(k,\omega)J_0(kr)kdk \\
RVF &= -\int_0^\infty F_{10}(k,\omega)J_1(kr)kdk \\
ZHF &= \int_0^\infty F_{11}(k,\omega)J_1(kr)kdk \\
RHF &= \int_0^\infty F_{12}(k,\omega)J_0(kr)kdk \\
&\quad - \frac{1}{r}\int_0^\infty \left[F_{12}(k,\omega) \;+\; F_{15}(k,\omega)\right] J_1(kr)dk \\
THF &= -\int_0^\infty F_{15}(k,\omega)J_0(kr)kdk \\
&\quad + \frac{1}{r}\int_0^\infty \left[F_{12}(k,\omega) \;+\; F_{15}(k,\omega)\right] J_1(kr)dk
\end{aligned}
$$

where the F_i functions here are solutions of a particular differential equation in z. Explicit expressions for the F_i exist for a uniform, isotropic wholespace and halfspace.

As an example, the *ZDD* for a source at $z =$ in a uniform, isotropic whole space is

$$ZDD = ZDD(r, z, 0, \omega) = \frac{-1}{4\pi\rho(i\omega)^2}\left[3\frac{\partial^3 F_\alpha}{\partial z^3} + k_\alpha^2\frac{\partial F_\alpha}{\partial z} - 3\frac{\partial^3 F_\beta}{\partial z^3} - 3k_\beta^2\frac{\partial F_\beta}{\partial z}\right]$$

where ρ is the density, α and β are the P- and S-wave velocities, respectively, and the F_V is the Sommerfeld integral

$$F_V = \frac{1}{R}e^{\frac{-i\omega R}{V}} = \int_0^\infty \frac{k}{\nu_V}e^{-\nu_V|z|}J_0(kr)dk$$

where $R^2 = r^2 + z^2$, $\nu_V^2 = k^2 - (\omega/V)^2$. Using the Sommerfeld integral and its derivatives, the F_1 for with *ZDD* is

$$F_1(k,\omega) = -\frac{1}{4\pi\rho(i\omega)^2}\left[(2k_\alpha^2 - 3k^2)e^{-\nu_\alpha|z|} + 3k^2 e^{-\nu_\beta|z|}\right] sgn(z)$$

with the function $sgn(z)$, defined as

$$sgn(z) = \begin{cases} -1 & z < 0 \\ 0 & z = 0 \\ 1 & z > 0 \end{cases}.$$

It can be shown (see Appendix) that the permanent offset in a uniform, isotropic wholespace due to a unit step in force or moment acting at $z = 0$ is obtained by evaluating the integrals after taking the limit $\omega \rightarrow 0$ of the F_i. In the case of *ZDD*, one has

$$F_1 = \frac{1}{4\pi\rho}\left[\frac{2}{\alpha^2} - \frac{3}{2}k|z|\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)\right] sgn(z)e^{-k|z|}$$

from which one obtains

$$ZDD = \frac{1}{4\pi\rho}\left[\frac{2}{\alpha^2}\frac{z}{R^3} - \frac{3}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)\left(\frac{3z^3}{R^5} - \frac{z}{R^3}\right)\right]$$

In the discussion that follows on the numerical implementation of the Fourier-Bessel transforms, the value of F_j functions as $\omega \rightarrow 0$, or equivalently when $k \gg \omega/V_{min}$, will be useful. The dependence of these integrands for large k will be guided by the $F_j(k, \omega = 0)$ functions for the cases of a uniform, isotropic wholespace and halfspace.

The integrands for a uniform isotropic wholespace are

$$F_1 = \frac{1}{4\pi\rho}\left[\frac{2}{\alpha^2} - \frac{3}{2}k|z|\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)\right] sgn(z)e^{-k|z|}$$

$$F_2 = -\frac{1}{4\pi\rho}\left[\frac{2}{\alpha^2} - \frac{3}{2}k|z|\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) - \frac{3}{2}\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\right] e^{-k|z|}$$

$$F_3 = -\frac{1}{4\pi\rho}\left[\frac{1}{\beta^2} + k|z|\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) - \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\right] e^{-k|z|}$$

$$F_4 = \frac{1}{4\pi\rho}\left[k|z|\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) + \frac{1}{\beta^2}\right] e^{-k|z|}sgn(z)$$

$$F_5 = -\frac{k}{4\pi\rho}\, e^{-k|z|}\frac{z}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)$$

$$F_6 = \frac{1}{4\pi\rho}\, e^{-k|z|}\left[\frac{k|z|}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) + \frac{1}{2}\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\right]$$

$$F_7 = \frac{1}{4\pi\rho\alpha^2}\, e^{-k|z|}sgn(z)$$

$$F_8 = -\frac{1}{4\pi\rho\alpha^2}\, e^{-k|z|}$$

$$F_9 = -\frac{1}{4\pi\rho k}\, e^{-k|z|}\left[\frac{k|z|}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) - \frac{1}{2}\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\right]$$

$$F_{10} = \frac{1}{4\pi\rho}\, e^{-k|z|}\frac{z}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)$$

$$F_{11} = -\frac{1}{4\pi\rho}\, e^{-k|z|}\frac{z}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)$$

$$F_{12} = \frac{1}{4\pi\rho k}\, e^{-k|z|}\left[\frac{k|z|}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) + \frac{1}{2}\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\right]$$

$$F_{13} = -\frac{1}{4\pi\rho\beta^2}\, e^{-k|z|}sgn(z)$$

$$F_{14} = -\frac{1}{4\pi\rho\beta^2}\, e^{-k|z|}$$

$$F_{15} = -\frac{1}{4\pi\rho\beta^2 k}\, e^{-k|z|}$$

and the integrands at the surface of a uniform, isotropic halfspace are

$$F_1 = \frac{\frac{\beta^2}{\alpha^2} - 3kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2\left(\frac{\beta^2}{\alpha^2} - 1\right)} e^{-kh}$$

$$F_2 = -\frac{\left(3 - 4\frac{\beta^2}{\alpha^2}\right) + 3kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2\left(\frac{\beta^2}{\alpha^2} - 1\right)} e^{-kh}$$

$$F_3 = \frac{2kh}{4\pi\rho\beta^2} e^{-kh}$$

$$F_4 = -\frac{2}{4\pi\rho\beta^2}(1 - kh) e^{-kh}$$

$$F_5 = -\frac{\frac{\beta^2}{\alpha^2} + kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2\left(\frac{\beta^2}{\alpha^2} - 1\right)} e^{-kh}$$

$$F_6 = -\frac{1 + kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2\left(\frac{\beta^2}{\alpha^2} - 1\right)} e^{-kh}$$

$$F_7 = \frac{2}{4\pi\rho(\beta^2 - \alpha^2)} e^{-kh}$$

$$F_8 = \frac{2}{4\pi\rho(\beta^2 - \alpha^2)} e^{-kh}$$

$$F_9 = -\frac{1 - kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2 k\left(\frac{\beta^2}{\alpha^2} - 1\right)} e^{-kh}$$

$$F_{10} = \frac{-\frac{\beta^2}{\alpha^2} + kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2 k\left(\frac{\beta^2}{\alpha^2} - 1\right)} e^{-kh}$$

$$F_{11} = -\frac{\frac{\beta^2}{\alpha^2} + kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2 k\left(\frac{\beta^2}{\alpha^2} - 1\right)} e^{-kh}$$

$$F_{12} = -\frac{1 + kh\left(\frac{\beta^2}{\alpha^2} - 1\right)}{4\pi\rho\beta^2 k\left(\frac{\beta^2}{\alpha^2} - 1\right)} e^{-kh}$$

$$F_{13} = \frac{2}{4\pi\rho\beta^2} e^{-kh}$$

$$F_{14} = -\frac{2}{4\pi\rho\beta^2} e^{-kh}$$

$$F_{15} = -\frac{2}{4\pi\rho\beta^2 k} e^{-kh}$$

An examination of these functions shows that most are of the form

$$(A + Bk)e^{-kh}$$

while two are of the form

$$(A + Bk)k^{-1}e^{-kh}$$

3 Numerical integration

Numerically, the integral must be approximated through a summation, but the real problem is that it is not feasible to approach the $k \to \infty$ limit and thus a k_{max} must used. Hence

$$\int_0^\infty F(k,\omega)J_n(kr)dk \approx \int_0^{k_{max}} F(k,\omega)J_n(kr)dk$$

For large k, e.g., $k \gg \omega/V_{min}$, the integrand should be of the form $[ak^{-1} + b + ck + dk^2]e^{-kh}J_n(kr)$. When h is large, the exponential decay means that it may be feasible to define a value for k_{max}. However, when h is small, then some other approach must be used.

One approach is to consider a function independent of frequency such that an analytic expression exists such that $g(r) = \int_0^\infty G(k)J_n(kr)dk$. Thus

$$\begin{aligned}\int_0^\infty F(k,\omega)J_n(kr)dk &= \int_0^\infty \Big[F(k,\omega) - G(k)\Big]J_n(kr)dk + g(r)\\ &\approx \int_0^{k_{max}} \Big[F(k,\omega) - G(k)\Big]J_n(kr)dk + g(r)\end{aligned}$$

Now if $G(k) \approx \lim_{\omega\to 0} F(k,\omega)$ for $k > k_{max}$, then the approximation above is effectively an equality.

The difficult part of imnplementing the numerical integration is when to use this asymptitic technique, e.g., whether to use the approximation

$$\int_0^\infty F(k,\omega)J_n(kr)dk \approx \sum_{i=0}^{N-1} F(k_i,\omega)J_n(kr)\Delta k)$$

or

$$\int_0^\infty F(k,\omega)J_n(kr)dk \approx \sum_{i=0}^{N-1} \Big[F(k_i,\omega) - G(k_i)\Big]J_n(kr)\Delta k + g(r)$$

where $\Delta k_i = 0.218\Delta k + i\Delta k$ and $\Delta k = 2\pi/L$ and L is chosed such that $L > 2r$ and $(L-r)^2 + z^2 > (V_{max}t_{max})^2$ (Bouchon, 1981) and Herrmann and Mandal (1986).

Subroutine *wvlimit* of program **hspec96strain** defines k_{max} and whether or not to use the asymptotic function $g(r) \leftrightarrow G(k)$ in the subroutine *wvlimit*.

If the asymptotic approach is used, then *solu* is used to define the parameters a, b and c in $[a + bk + ck^2]e^{-kh}$. In its current form (April 9, 2025), $[a + bk]e^{-kh}$ is used for the determination of displacements and the $\frac{d}{dr} derivative, and [bk + ck^2]e^{-kh}$ is used for the d$\frac{}{dz}$ derivatives.

For the F_j, $j = 9, 10, 12, 15$, which have 4 k in the denominator, the trick is that

$$\int_0^\infty F_j J_n(kr)dk = \int_0^\infty [pkF_j]\ [J_n(kr)/k]dk \tag{1}$$

and thus at large k $kF_j(k)$ is modeled as $[a + bk]e^{-kh}$.

The logic in subroutine *wvlimit* is somewhat *ad hoc* and requires additional effort, especially in the case the $h \to 0$. It is not perfect, as seedn in the examples. **This is a research area.**

The next figure illustrates the region in (k,ω) space for F_2 for a uniform, isotropic halfspace that is used in the numerical integration. Specifically, in the case of $h = 1$km and $z = 0$km, the asymptotic approach is used.

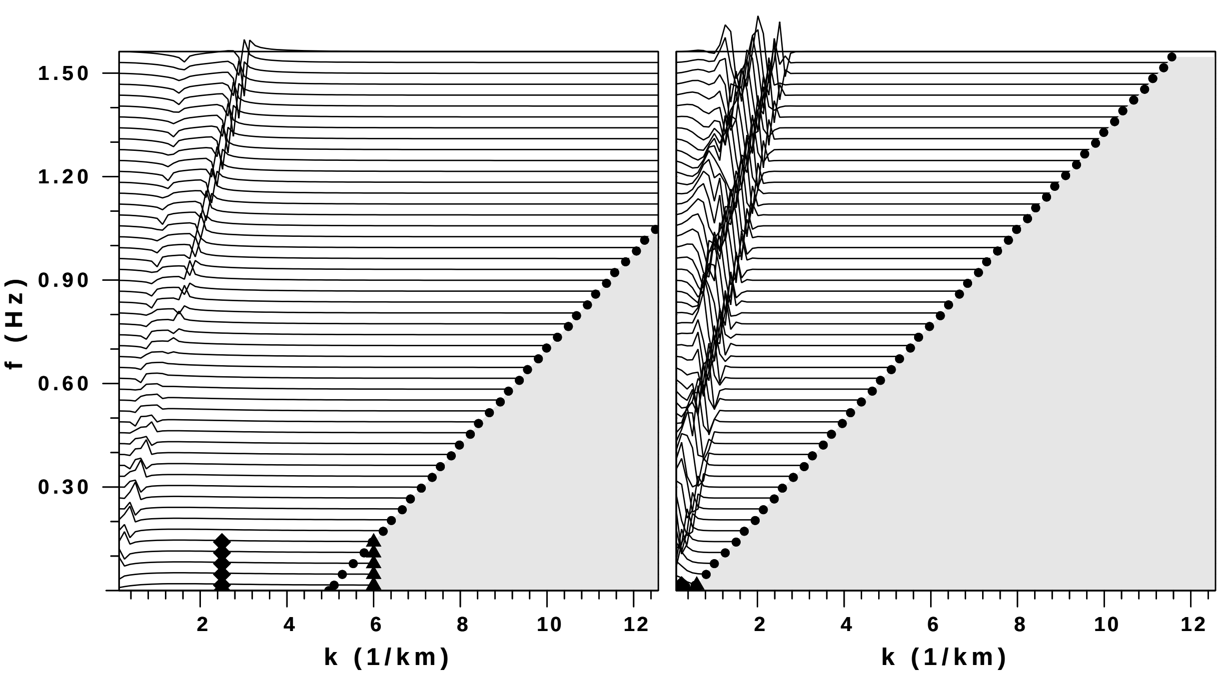

Figure 1: The real part of the $F_2(k,f)$ integrand for source depths of 1.0 (left) and 10.0 km (right). The circles represent $4\omega/v_{min}$, while the diamonds and triangle represent wavenumbers $2.5/h$ and $6/h$, respectively when the asymptotic technique is applied. The shaded area is not used for the integration.

This figure also highlights the computational effort required to make synthetics. The integrand $F(k,\omega)$ is evaluated in a triangular region. The number of evaluations ois $N_k N_\omega$. The Nyquist frequency, defining the maximum frequency, is related to the sampling interval by $f_{max} = 1/2\Delta$, and the length of the time series is $N\Delta t$. hus

4 Bessel functions

Although the Bessel functions are intrinsic to today's FORTRAN compilers, special care must be taken when using them for determining Green's functions that have near-field terms, e.g., RDS, TDS, RSS, TSS, RHF and THF because of the need to evaluate the the $J_1(kr)/r$ and $J_2(kr)/r$ terms as $r \rightarrow 0$. In addition, computation of strain requires the derivatives $dJ_n(kr)/dr$ and $d[J_n(kr)/r)]/dr$ which are not in standard function libraries.

Some Bessel function properties (Abramowitz and Stegun, AS9.1.27 and AS9.1.28) are

$$
\begin{aligned}
J_2(kr) &= \frac{2}{kr} J_1(kr) - J_0(kr) \\
\frac{dJ_0(kr)}{dr} &= -kJ_1(kr) \\
\frac{dJ_1(kr)}{dr} &= kJ_0(kr) - \frac{1}{r} J_1(kr) \\
\frac{dJ_2(kr)}{dr} &= kJ_1(kr) - \frac{2}{r} J_2(kr) \\
\frac{d[J_1(kr)/r]}{dr} &= \frac{1}{r}\Big[\frac{dJ_1(kr)}{dr} - \frac{1}{r} J_1(kr)\Big] \\
\frac{d[J_2(kr)/r]}{dr} &= \frac{1}{r}\Big[\frac{dJ_2(kr)}{dr} - \frac{1}{r} J_2(kr)\Big]
\end{aligned}
$$

where the argument kr is used to bring the relations into the form required for wavenumber integreation.

Obviously there is an apparent problem with the $\frac{1}{r}$ terms at $r = 0$, but there can also be numerical problems when r is small. The subroutine *dhank* in **hspec96strain** uses the series expansions of (Abramowitz and Stegun §9.4). These are summarized here. Since this section focuses on the evaluation of the functions and their derivatives for small r, those cases are given last:

4.1 $3 \leq x < \infty$

$$
\begin{aligned}
J_0(x) &= x^{-\frac{1}{2}} f_0 \cos\theta_0 \\
J_1(x) &= x^{-\frac{1}{2}} f_1 \cos\theta_1
\end{aligned}
$$

where the f_0 and θ_0 are given in (Abramowitz and Stegun, 9.4.3) and the f_1 and θ_1 in (9.4.6). Since x is never zero, the relations given above can be used to obtain the required dreivatives.

4.2 $0 \leq x \leq 3$

In this case, one could carefully use an *if* statement in the code to flag the $r = 0$ case, but there will still be a numerical roundoff problem that will affect the synthetics. Instead the code starts with the polynomial approximations for $J_0(z)$ (AS 9.4.1) and $x^{-1}J_1(x)$ (AS 9.4.4). Rather than clutter this discussion with actual coeffients, these polynomials are of the form

$$
\begin{aligned}
J_0(x) &= a_0 + a_2x^2 + a_4x4 + \mathrm{a}_6x^6 + a_8x^8 + a_{10}x^{10} + a_{12}x^{12} \qquad units \\
J1Z(x) = x^{-1}J_1(x) &= b_0 + b_2x^2 + b_4x^4 + b_6x^6 + b_8x^8 + b_{10}x^{10} + b_{12}x^{12}
\end{aligned}
$$

(Note that Abramowitz and Stegun use a power series in terms of $(x/3)$.)

Given these coefficients, one can write

$$
J_2(x) = c_2x^2 + c_4x^4 + c_6x^6 + c_8x^8 + c_{10}x^{10} + c_{12}x^{12}
$$

where

$$
\begin{aligned}
c_0 &= 0 \text{ using actual } a_0 \text{ and } b_0 \text{coefficients}\\
c_2 &= 2b_2 - a_2\\
c_4 &= 2b_4 - a_4\\
c_6 &= 2b_6 - a_6\\
c_8 &= 2b_8 - a_8\\
c_{10} &= 2b_{10} - a_{10}\\
c_{12} &= 2b_{12} - a_{12}
\end{aligned}
$$

Given these we have

$$
\begin{aligned}
J_1(kr) &= kr\ J1Z(kr)\\
\frac{1}{r}J_1(kr) &= k\ J1Z(kr)\\
\frac{dJ_0(kr)}{dr} &= -J_1(kr)\\
\frac{dJ_1(kr)}{dr} &= k\Big[b_0 + 3b_2x^2 + 5b_4x^4 + 7b_6x^6 + 9b_8x^8 + 11b_{10}x^{10} + 13b_{12}x^{12}\Big]_{x=kr}\\
\frac{d[J_1(kr)/r]}{dr} &= k^2\Big[2b_2x + 4b_3x^3 + 6B_6x^5 + 8b_8x^7 + 10b_{10}x^9 + 12b_{12}x^{11}\Big]_{x=kr}\\
\frac{dJ_2(kr)}{dr}) &= k\Big[2c_2x + 4c_4x^3 + 6c_6x^5 + 8c_8x^7 + 10c_{10}x^9 + 12c_{12}x^{11}\Big]_{x=kr}\\
\frac{d[J_2(kr)/r]}{dr}) &= k^2\Big[c_2 + 3c_4x^2 + 5c_6x^4 + 7c_8x^6 + 9c_{10}x^9 + 11c_{12}x^{10}\Big]_{x=kr}
\end{aligned}
$$

Finally, in the actual code, the series are evaluated using Horner's rule, e.g., $y = a_0+x(a_1+x*(a_2))$ rather than $y = a_0+a_1x+a_2x^2$ for numerical and computational efficiency.

5 References

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6 Appendix

For a step source-time function, the displacement traces for the complete solution have a step offset at times following the S-wave arrival. These offsets are the permanent deformation due to the application of a step force. This can easily be obtained by taking the limit of the Fourier transform of the displacement as $\omega \rightarrow 0$. The proof for this statement is very simple. If $d(t)$ is the displacement as a function of t, it is related to the velocity, $v(t)$, through the integral definition $d(t) = \int_{-\infty}^{t} v(\tau)d\tau$. Thus the permanent deformation is $d(\infty)$. In addition the Fourier transform of the velocity at zero frequency is $V(0) = \int_{-\infty}^{\infty} v(\tau)d\tau$, where we use the Fourier transform pair $v(t) \Leftrightarrow V(\omega)$. Thus the permanent offset is related to the $V(\omega)$ by the relation $d(\infty) = V(0)$.